AN ALGORITHM FOR THE TOPOLOGICAL DEGREE
OF A MAPPING IN $n$-SPACE$^1$

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1. Introduction. In this paper we announce a new formula for computing the topological degree $d(F, P, \theta)$, where $F = (f^1, \cdots, f^n)$ is a vector of real continuous functions mapping a polyhedron $P$ in $\mathbb{R}^n$ into $\mathbb{R}^n$, and $\theta$ is the zero vector in $\mathbb{R}^n$.

Let $A = [a_{ij}]$ be an $n \times n$ real matrix, and let $A_i$ denote the $i$th row of $A$. We use the convenient notation $\Delta_n(A_1, \cdots, A_n)$ for the determinant of $A$, and $|A_i| \equiv (a_{i1}^2 + \cdots + a_{in}^2)^{1/2}$ for the Euclidean norm of $A_i$.

Let $X_0, X_1, \cdots, X_q$ denote $q + 1$ points in $\mathbb{R}^n$, where $q \leq n$, such that the vectors $X_i - X_0$, $i = 1, 2, \cdots, q$, are linearly independent. A $q$-simplex with vertices at $X_0, \cdots, X_q$ is defined by

$$S_q(X_0, \cdots, X_q) \equiv \left\{ X \in \mathbb{R}^n : X = \sum_{i=0}^{q} \lambda_i X_i, \lambda_i \geq 0, \sum_{i=0}^{n} \lambda_i = 1 \right\}. $$

We denote by $[X_0 \cdots X_q]$ the oriented $q$-simplex, defined as in [2]. For example, if $q = n$, then $[X_0 \cdots X_q] = [X_0 \cdots X_n]$ is said to be positively (negatively) oriented in $\mathbb{R}^n$ if $\Delta_{n+1}(Z_0, \cdots, Z_n) > 0 (< 0)$, where $Z_i = (1, X_i)$.

Let $P$ be a connected, $n$-dimensional closed polyhedron represented as a "sum" of $m'$ positively oriented $n$-simplexes in the form

$$P = \sum_{j=1}^{m'} [X_0^{(j)} \cdots X_n^{(j)}]$$

such that the intersection of any two of the simplexes has zero $n$-dimensional volume.


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The boundary of \([X_0 \cdots X_n]\) is represented in terms of oriented \(n-1\)-simplexes by

\[
b[X_0 \cdots X_n] = \sum_{i=0}^{n} (-1)^i [X_0 \cdots \hat{X}_i \cdots X_{i+1} \cdots X_n]
\]

(see [2]). By means of this expansion, the boundary of \(P\) may be represented in the form

\[
b(P) = \sum_{j=1}^{m} t_j [Y_1^{(j)} \cdots Y_n^{(j)}]
\]

where \(P\) is defined in (1.1), and \(t_j = \pm 1\). For example, if \(n = 1\),

\[
P = [X_0 X_1] + [X_1 X_2] + \cdots + [X_{m-1} X_m],
\]

\[
b(P) = [X_m] - [X_0].
\]

Let \(F\) be a vector of \(n\) real \(C^1\) functions defined on \(P\), such that \(F \neq \theta = (0, \cdots, 0)\) on \(b(P)\). We denote by \(d(F, P, \theta)\) the topological degree of \(F\) at \(\theta\) relative to \(P\). We define \(d(F, P, \theta)\) by

\[
d(F, P, \theta) = \frac{1}{2} \left\{ \frac{F(X_m)}{|F(X_m)|} - \frac{F(X_0)}{|F(X_0)|} \right\}
\]

if \(n = 1\),

\[
d(F, P, \theta) = \frac{1}{\Omega_{n-1}} \int_{b(P)} \frac{1}{|F|^n} \Delta_n \left( F, \frac{\partial F}{\partial u_1}, \cdots, \frac{\partial F}{\partial u_{n-1}} \right) du_1 \cdots du_{n-1}
\]

if \(n > 1\),

where \(\Omega_{n-1}\) denotes the \(n-1\) dimensional volume of the surface of the \(n\)-sphere, and where \(F = F(X(U))\) is suitably parametrized as a function of \(U = (u_1, \cdots, u^{n-1})\) (see [1, pp. 465-467]). If \(F\) is merely real and continuous on \(P\), but not necessarily of class \(C^1\), we define \(d(F, P, \theta)\) by

\[
d(F, P, \theta) = \lim_{\nu \to \infty} d(F^{(\nu)}, P, \theta),
\]

where \(F^{(\nu)}\) is real and of class \(C^1\) on \(P\) for \(\nu = 1, 2, \cdots, \max_{X \in P} |F(X) - F^{(\nu)}(X)| \to 0\) as \(\nu \to \infty\), and \(d(F^{(\nu)}, P, \theta)\) is defined by means of (1.4).

The integral formula (1.4) is due to Kronecker [1, pp. 465-467].

Another integral for \(d(F, P, \theta)\) has been given by Heinz [3]. In the following section we shall describe another procedure for evaluating \(d(F, P, \theta)\), which depends only on the sign of the components of \(F\) at a finite number of points of \(b(P)\).
2. Formula for \( d(F, P, \theta) \). If \( a \) is a real number, we define \( \text{sgn} \ a \) by
\[
\text{sgn} \ a = -1, 0 \text{ or } 1 \text{ if } a < 0, = 0 \text{ or } > 0 \text{ respectively.}
\]
We define \( \text{sgn} \ F \) by \( \text{sgn} \ F = (\text{sgn} \ f^1, \cdots, \text{sgn} \ f^n) \). Let us set
\[
(2.1) \quad \delta_m(F, P, \theta) = \frac{1}{2^n n!} \sum_{j=1}^{m} t_j \Delta_n(\text{sgn} F(Y_1^{(j)}), \cdots, \text{sgn} F(Y_n^{(j)}))
\]
where the \( t_j \) and \( Y_i^{(j)} \) are the same as in (1.2). This formula is used to compute \( d(F, P, \theta) \) by means of the following

**Algorithm 2.1.**

1. Let \( p \) be a fixed positive integer.
2. Set \( \delta = \delta_m(F, P, \theta) \) as defined in (2.1).
3. Revise the definition of \( b(P) \) as follows: For \( j = 1, 2, \cdots, m, \)
   a. locate the longest segment \( Y_k^{(j)}Y_i^{(j)} \) \((k < 1)\) of the oriented simplex \( t_j[Y_i^{(j)} \cdots Y_n^{(j)}] \) in (1.2), and set \( A = (Y_k^{(j)} + Y_i^{(j)})/2; \)
   b. replace \( t_j[Y_i^{(j)} \cdots Y_n^{(j)}] \) according to:
\[
t_j[Y_1^{(j)} \cdots Y_k^{(j)} \cdots Y_i^{(j)} \cdots Y_n^{(j)}] \leftarrow t_j[Y_1^{(j)} \cdots A \cdots Y_i^{(j)} \cdots Y_n^{(j)}],
\]
\[
t_{j+m}[Y_1^{(j+m)} \cdots Y_k^{(j+m)} \cdots Y_i^{(j+m)} \cdots Y_n^{(j+m)}] \leftarrow t_j[Y_1^{(j)} \cdots Y_k^{(j)} \cdots A \cdots Y_n^{(j)}];
\]
4. replace \( m \) by \( 2m \) to get a new decomposition of \( b(P) \) in terms of (twice as many) oriented simplexes.
5. Set \( e = \delta_m(F, P, \theta) \) as defined in (2.1), with the new \( b(P) \).
6. If \( \delta = e = \text{integer} \), go to Step 6. Otherwise set \( \delta = e \) and return to Step 3.
7. Replace \( p \) by \( p - 1 \). If the resulting \( p \) is positive, return to Step 3. Otherwise print out \( m, \delta \).

**Assumption 2.2.** Let \( F \) be continuous and real on \( P \), where \( P \) is defined as in equation (1.1). Let \( b(P) \) be defined as in equation (1.2), and let \( F \neq 0 \) on \( b(P) \). If \( n > 1 \), for all \( 1 < \mu \leq n, \phi^1 = f^i_l, j_k \neq j_l \) if \( k \neq l \), and \( \Phi_\mu = (\phi^1, \cdots, \phi^\mu) \), we assume that the sets \( T(A_\mu) = \{ X \in b(P); \Phi_\mu(X)/|\Phi_\mu(X)| = a_\mu \} \cap S_{\mu-1} \) and \( b(P) - T(A_\mu) \) consist of a finite number of connected subsets of \( b(P) \), for all vectors \( a_\mu = (\pm 1, 0, \cdots, 0), (0, \pm 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, \pm 1), \) and for all \( \mu - 1 \)-simplexes \( S_{\mu-1} \) on \( b(P) \).

**Theorem 2.3.** If Assumption 2.2 is satisfied and if the integer \( p \) in Algorithm 2.2 is chosen sufficiently large, then Algorithm 2.1 prints out
finite integers \( m \) and \( \delta \), where \( \delta = d(F, P, \theta) \), and where \( P \) is defined in (1.1).

REFERENCES


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