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SOME SUBALGEBRAS OF $L^\infty(T)$ DETERMINED BY THEIR MAXIMAL IDEAL SPACES

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1. **Introduction.** Sarason [4], [5] has shown, by using the notions of asymptotic multiplicity and vanishing mean oscillation, that $H^\infty(T) + C$ is determined by its maximal ideal space. In this note we announce a generalization of this result to include various superalgebras of $H^\infty(T) + C$. As intermediate steps, we develop localized notions of asymptotic multiplicity and VMO.

2. **Definitions and notation.** (a) Let

$$G_{n,\lambda} = \{z\colon 1 - 1/n < |z| < 1, |\arg z - \arg \lambda| < 1/n\}$$

for $\lambda \in T$, $n = 1, 2, \cdots$. For a closed subalgebra A of $L^\infty(T)$ containing $H^\infty(T)$, the Poisson integral is said to be *asymptotically multiplicative* on A at λ if, for $f, g \in A$, $\epsilon > 0$, there exists an N such that $|\hat{f}(z)\hat{g}(z) - \widehat{fg}(z)| < \epsilon$ for $z \in G_{n,\lambda}$ for all $n \geqslant N$.

(b) Now let I be an arc on T; we define θ_I and r_I such that

(i) $e^{i\theta_I}$ is the center of I, and

(ii) $r_I = 1 - \pi m(I)$.

Now we define a collection of arcs $J_{n,\lambda} = \{\text{subarcs } I \text{ of } T\colon r_I e^{i\theta_I} \in G_{n,\lambda}\}$, and for $f \in L^1(T)$ we define

$$M_{n,\lambda}(f) = \operatorname*{Sup}_{I \in J_{n,\lambda}} \frac{1}{m(I)} \int_I |f - f_I|\, dm.$$

We say that $f \in \mathrm{VMO}_\lambda$ if $f \in \mathrm{BMO}$ and $\lim_{n\to\infty} M_{n,\lambda}(f) = 0$. See [4] for a definition and discussion of BMO.

(c) Let $E \subseteq T$; then $L_E^\infty(T)$ will denote the set of functions in $L^\infty(T)$ which can be extended continuously on the set E. When E is a singleton, say $E = \{\lambda\}$, $L_E^\infty(T)$ will be denoted $L_\lambda^\infty(T)$. In case E is σ-compact, it is known [2] that $H^\infty(T) + L_E^\infty(T)$ is a closed algebra.

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(d) We will also be concerned with the algebra B_λ, defined to be the closed subalgebra of $L^\infty(T)$ generated by $H^\infty(T)$, and those functions on T continuous except possibly at λ and having two-sided limits at λ.

(e) For a closed subalgebra A of $L^\infty(T)$ which contains $H^\infty(T)$, we denote the maximal ideal space of A by $M(A)$, and we denote by Y_λ the fibre over λ of the maximal ideal space of $H^\infty(T)$; see [3].

3. **Main results.**

THEOREM 1. *Let A be a closed subalgebra of $L^\infty(T)$, which contains $H^\infty(T)$; then the following are equivalent:*

(i) $Y_\lambda \subseteq M(A)$.

(ii) *The Poisson integral is asymptotically multiplicative on A at λ.*

THEOREM 2. *Let $w \in L^\infty(T), \lambda \in T$ such that*

(i) $|w(e^{i\theta})| = 1$ *a.e.,*

(ii) $|\hat{w}(z)|$ *is continuous at* λ;

then $w \in VMO_\lambda \cap L^\infty(T)$.

The next three theorems concern algebras determined by their maximal ideal spaces.

THEOREM 3. *Let A be a closed subalgebra of $L^\infty(T)$. If*

(i) $H^\infty(T) + L^\infty_\lambda(T) \subseteq A$ *and*

(ii) $M(H^\infty(T) + L^\infty_\lambda(T)) = M(A)$,

then $H^\infty(T) + L^\infty_\lambda(T) = A$.

Using Theorem 3 and some results of Davie, Gamelin and Garnett [2], we show

THEOREM 4. *Let A be a closed subalgebra of $L^\infty(T)$, and let E be a σ-compact subset of T. If*

(i) $H^\infty(T) + L^\infty_E(T) \subseteq A$ *and*

(ii) $M(H^\infty(T) + L^\infty_E(T)) = M(A)$,

then $H^\infty(T) + L^\infty_E(T) = A$.

THEOREM 5. *Let A be a closed subalgebra of $L^\infty(T)$. If* (i) $B_1 \subseteq A$ *and* (ii) $M(B_1) = M(A)$, *then* $B_1 = A$.

4. **Remark.** Two students of Sarason have independently demonstrated at least some of these results. Sheldon Axler [1] has proved Theorem 4 and Alice Chang has proved Theorem 5.

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