A COMPARATIVE STUDY OF THE ZEROS OF DIRICHLET $L$-FUNCTIONS

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We give a comparative study of the zeros of Dirichlet $L$-functions.
Details will appear later.

1. Let $\chi_1$ and $\chi_2$ be distinct primitive characters of the same modulus $q$, and let $L(s, \chi_i)$, for $i = 1, 2$, be the corresponding Dirichlet $L$-functions.
It is quite natural to guess that $L(s, \chi_1)$ and $L(s, \chi_2)$ have no coincident zero. In other words even a single zero will determine a Dirichlet $L$-function, or more generally, a “zeta-function”. To be more precise, we call $\rho$ a coincident zero of $L(s, \chi_1)$ and $L(s, \chi_2)$ if $L(\rho, \chi_1) = L(\rho, \chi_2) = 0$ with the same multiplicities. And we call $\rho$ a noncoincident zero if it is not coincident. Then we can show

**Theorem 1.** Let $\chi_1$ and $\chi_2$ be distinct primitive characters of the same modulus. Then a positive proportion of the zeros of $L(s, \chi_1)$ and $L(s, \chi_2)$ are noncoincident.

Next, it is quite natural to guess that the distribution of the zeros of $L(s, \chi_1)$ and $L(s, \chi_2)$ are independent. To state our results, let $\gamma_n(\chi)$ be the ordinate of the $n$th zero of $L(s, \chi)$ such that $0 \leq \gamma_n(\chi) \leq \gamma_{n+1}(\chi)$. Further we define $\gamma_n(\chi_1) \leq \gamma_m(\chi_2)$ if $\gamma_n(\chi_1) < \gamma_m(\chi_2)$, and $\tau_n(\chi_1) \leq \gamma_m(\chi_2) \leq \gamma_{n+1}(\chi_1) \leq \gamma_{m+1}(\chi_2) \leq \cdots$ if $\gamma_n(\chi_1) = \gamma_{n+1}(\chi_1) = \cdots = \gamma_m(\chi_2) = \gamma_{m+1}(\chi_2) = \cdots$. Then we get

**Theorem 2.** Under the same hypothesis as above, for a positive proportion of $\gamma_n(\chi_1)$'s, there does not exist a $\gamma(\chi_2)$ for which $\gamma_n(\chi_1) < \gamma(\chi_2) < \gamma_{n+1}(\chi_1)$.

Further we define $\Delta_n(\chi_1, \chi_2)$ to be $n - m$ if $\gamma_m(\chi_1) \leq \gamma_n(\chi_2)$. 


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THEOREM 3. For any positive increasing function $\Phi(n)$ which tends to $\infty$ as $n$ tends to $\infty$, we have

$$|\Delta_n(x_1, x_2)| > 2\pi(\log \log n)^{1/2}/\Phi(n)$$

for almost all $n$. In particular, $\gamma_n(x_2)$ almost never satisfies $\gamma_n(x_1) \leq \gamma_n(x_2) \leq \gamma_{n+1}(x_1)$.

Theorems 1 and 2 come from a mean value theorem about

$$\int_0^T (S(t + h, x_1) - S(t, x_1) - (S(t + h, x_2) - S(t, x_2))^d dt,$$

where $S(t, \chi) = \pi^{-1} \arg L(\frac{1}{2} + it, \chi)$ as before (cf. [1]). Theorem 3 comes from a mean value theorem about $\int_0^T (S(t, x_1) - S(t, x_2))^d dt$. If we use mean value theorems about

$$\sum' \sum' (S(t + h, x_1) - S(t, x_1) - (S(t + h, x_2) - S(t, x_2))^d$$

and

$$\sum' \sum' (S(t, x_1) - S(t, x_2))^d,$$

where in the summation $\chi_i$ runs over all nonprincipal characters of modulus $q$ for each $i = 1, 2$, then we get $q$-analogues of our theorems.

2. As an application of our methods we can get some results about a problem of Knapowski-Turán. Let $q$ be a given fixed positive integer. Assume that $(b, q) = (d, q) = 1$ and $b \equiv d \pmod q$. Let $\chi$ be a character of modulus $q$. We write $g(\chi) = (\chi(b) - \chi(d))/\varphi(q)$, and $\mu(\rho) = \mu_{b,d}(\rho) = \sum_x g(\chi)m_x(\rho)$, where $\chi$ runs over all characters of modulus $q$ and $m_x(\rho)$ is the multiplicity of $\rho$ as a zero of the Dirichlet $L$-functions $L(s, \chi)$.

Knapowski and Turán proposed the following problem in their study of prime numbers:

Estimate $f(T) = \sum_{0 < \text{Im} \rho < T, \mu(\rho) \neq 0} 1$ (cf. [3]). Concerning this problem, Kátai (unpublished) and Grosswald [2] proved independently the existence of infinitely many $\rho$'s with $\mu(\rho) \neq 0$. Later Turán obtained the following results (cf. [6]).

(1) For $T > \psi(q)$ we have the inequality $f(T) > c_1 \exp((\log T)^{1/5})$. 

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(2) Under the assumption of the generalized Riemann hypothesis we have \( f(T) > C_2 T^{1/2} \) for \( T > \psi(q) \), where the \( C_\nu \) are numerical constants and \( \psi(q) \) is an explicit function of \( q \). Recently Motohashi [4] obtained the following results.

(1) For \( T > \psi(q) \) we have \( f(T) > T^{1/10} \left( \log T \right)^{-3} \).

(2) For any sufficiently large \( T \) there exists at least one \( q \) with \( \frac{1}{2} T^{1/2} \left( \log T \right)^{-51} < q < T^{1/2} \left( \log T \right)^{-51} \) such that \( f(T) > T^{3/28} \left( \log T \right)^{-45} \).

Now we can show

**Theorem 4.** For \( T > \psi(q) \) we have \( f(T) > A T \log T \), where \( \psi(q) \) is some explicit function of \( q \) and the positive constant \( A \) may depend on \( q \).

In fact, we can take \( \psi(q) = \exp(\exp(C_1 q)) \) and \( A = \exp(-C_2 q) \) with suitable positive absolute constants \( C_1 \) and \( C_2 \).

We prove this from a mean value theorem concerning

\[
\int_0^T \left| \sum_{\chi} g(\chi) S(t + h, \chi^*) - S(t, \chi^*) \right|^2 dt,
\]

where \( \chi^* \) is the primitive character attached to \( \chi \).

**References**


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