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A COMPARATIVE STUDY OF THE ZEROS OF DIRICHLET L-FUNCTIONS

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We give a comparative study of the zeros of Dirichlet L-functions. Details will appear later.

1. Let χ_1 and χ_2 be distinct primitive characters of the same modulus q, and let $L(s, \chi_i)$, for $i = 1, 2$, be the corresponding Dirichlet L-functions. It is quite natural to guess that $L(s, \chi_1)$ and $L(s, \chi_2)$ have no coincident zero. In other words even a single zero will determine a Dirichlet L-function, or more generally, a "zeta-function". To be more precise, we call ρ a coincident zero of $L(s, \chi_1)$ and $L(s, \chi_2)$ if $L(\rho, \chi_1) = L(\rho, \chi_2) = 0$ with the same multiplicities. And we call ρ a noncoincident zero if it is not coincident. Then we can show

THEOREM 1. *Let χ_1 and χ_2 be distinct primitive characters of the same modulus. Then a positive proportion of the zeros of $L(s, \chi_1)$ and $L(s, \chi_2)$ are noncoincident.*

Next, it is quite natural to guess that the distribution of the zeros of $L(s, \chi_1)$ and $L(s, \chi_2)$ are independent. To state our results, let $\gamma_n(\chi)$ be the ordinate of the nth zero of $L(s, \chi)$ such that $0 \leqslant \gamma_n(\chi) \leqslant \gamma_{n+1}(\chi)$. Further we define $\gamma_n(\chi_1) \leqslant \gamma_m(\chi_2)$ if $\gamma_n(\chi_1) < \gamma_m(\chi_2)$, and $\gamma_n(\chi_1) \leqslant \gamma_m(\chi_2) \leqslant \gamma_{n+1}(\chi_1) \leqslant \gamma_{m+1}(\chi_2) \leqslant \cdots$ if $\gamma_n(\chi_1) = \gamma_{n+1}(\chi_1) = \cdots = \gamma_m(\chi_2) = \gamma_{m+1}(\chi_2) = \cdots$. Then we get

THEOREM 2. *Under the same hypothesis as above, for a positive proportion of $\gamma_n(\chi_1)$'s, there does not exist a $\gamma(\chi_2)$ for which $\gamma_n(\chi_1) \leqslant \gamma(\chi_2) \leqslant \gamma_{n+1}(\chi_1)$.*

Further we define $\Delta_n(\chi_1, \chi_2)$ *to be* $n - m$ *if* $\gamma_m(\chi_1) \leqslant \gamma_n(\chi_2) \leqslant$

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$\gamma_{m+1}(\chi_1)$. Then we can show

THEOREM 3. *For any positive increasing function* $\Phi(n)$ *which tends to* ∞ *as* n *tends to* ∞, *we have*

$$|\Delta_n(\chi_1, \chi_2)| > 2\pi(\log\log n)^{1/2}/\Phi(n)$$

for almost all n. *In particular,* $\gamma_n(\chi_2)$ *almost never satisfies* $\gamma_n(\chi_1) \leqslant \gamma_n(\chi_2) \leqslant \gamma_{n+1}(\chi_1)$.

Theorems 1 and 2 come from a mean value theorem about

$$\int_0^T \{S(t+h, \chi_1) - S(t, \chi_1) - (S(t+h, \chi_2) - S(t, \chi_2))\}^l dt,$$

where $S(t, \chi) = \pi^{-1}\arg L(\tfrac{1}{2} + it, \chi)$ as before (cf. [1]). Theorem 3 comes from a mean value theorem about $\int_0^T (S(t, \chi_1) - S(t, \chi_2))^l dt$. If we use mean value theorems about

$$\sum_{\chi_1}{}' \sum_{\chi_2}{}' \{S(t+h, \chi_1) - S(t, \chi_1) - (S(t+h, \chi_2) - S(t, \chi_2))\}^l$$

and

$$\sum_{\chi_1}{}' \sum_{\chi_2}{}' \{S(t, \chi_1) - S(t, \chi_2)\}^l,$$

where in the summation χ_i runs over all nonprincipal characters of modulus q for each $i = 1, 2$, then we get q-analogues of our theorems.

2. As an application of our methods we can get some results about a problem of Knapowski-Turán. Let q be a given fixed positive integer. Assume that $(b, q) = (d, q) = 1$ and $b \not\equiv d \pmod q$. Let χ be a character of modulus q. We write $g(\chi) = (\bar\chi(b) - \bar\chi(d))/\varphi(q)$, and $\mu(\rho) = \mu_{b,d}(\rho) = \Sigma_\chi g(\chi) m_\chi(\rho)$, where χ runs over all characters of modulus q and $m_\chi(\rho)$ is the multiplicity of ρ as a zero of the Dirichlet L-functions $L(s, \chi)$. Knapowski and Turán proposed the following problem in their study of prime numbers:

Estimate $f(T) = \Sigma_{0<\operatorname{Im}\rho<T; \mu(\rho)\neq 0} 1$ (cf. [3]). Concerning this problem, Kátai (unpublished) and Grosswald [2] proved independently the existence of infinitely many ρ's with $\mu(\rho) \neq 0$. Later Turán obtained the following results (cf. [6]).

(1) For $T > \psi(q)$ we have the inequality $f(T) > c_1 \exp((\log T)^{1/5})$.

(2) Under the assumption of the generalized Riemann hypothesis we have $f(T) > C_2 T^{1/2}$ for $T > \psi(q)$, where the C_ν are numerical constants and $\psi(q)$ is an explicit function of q. Recently Motohashi [4] obtained the following results.

(1) For $T > \psi(q)$ we have $f(T) > T^{1/10}(\log T)^{-3}$.

(2) For any sufficiently large T there exists at least one q with $\frac{1}{2}T^{1/2}(\log T)^{-51} \leqslant q \leqslant T^{1/2}(\log T)^{-51}$ such that $f(T) > T^{3/28}(\log T)^{-45}$.

Now we can show

THEOREM 4. *For $T > \psi(q)$ we have $f(T) > AT \log T$, where $\psi(q)$ is some explicit function of q and the positive constant A may depend on q.*

In fact, we can take $\psi(q) = \exp(\exp(C_1 q))$ and $A = \exp(-C_2 q)$ with suitable positive absolute constants C_1 and C_2.

We prove this from a mean value theorem concerning

$$\int_0^T \left| \sum_{\chi} g(\chi)(S(t+h, \chi^*) - S(t, \chi^*)) \right|^l dt,$$

where χ^* is the primitive character attached to χ.

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