 ULTRAFILTERS AND ALMOST DISJOINT SETS. II

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Unless otherwise stated, $\kappa$ is an arbitrary infinite regular cardinal. For every infinite cardinal $\kappa$, $\mu\kappa$ is the family of uniform ultrafilters on $\kappa$. Our main result is:

**Theorem 1.** Suppose that $2^\kappa = \kappa^+$. Then for every $U \in \mu\kappa$ there is a family $\{a_x : x \in U\}$ such that: for every $x \in U$, $a_x \subseteq x$ and $|a_x| = \kappa$; and for every $x, y \in U$ with $x \neq y$, $|a_x \cap a_y| < \kappa$.

This answers a question of Comfort communicated privately to the author and partially answers a question of Hindman [5]. It is still open whether Theorem 1 holds for singular $\kappa$ as well. The hypothesis $2^\kappa = \kappa^+$ cannot be outright removed, since by [1] it is consistent with ZFC that there is no $A \subseteq P(\kappa)$ such that $|A| = 2^\kappa$, $|a| = \kappa$ for every $a \in A$, and $|a \cap b| < \kappa$ for every $a, b \in A$ with $a \neq b$. See [4], [5] and [8] for other relevant results.

**Definition 1.** For $A \subseteq P(\kappa)$ and ideal $I \subseteq P(\kappa)$, $I$ is said to be dense in $A$ modulo sets of power $< \kappa$ if for each $x \in A$ such that $|x| = \kappa$, there is some $y \in I$ with $y \subseteq x$ and $|y| = \kappa$. For brevity we shall write "$I$ is dense in $A$ mod($< \kappa$)". $I$ is dense mod($< \kappa$) if $I$ is dense in $P(\kappa)$ mod($< \kappa$).

"$I$ is $\lambda$-complete" is defined as in [7].

**Theorem 2.** For every $U \in \mu\kappa$ there is a $\kappa$-complete ideal $I \subseteq P(\kappa) - U$ such that $I$ is dense mod($< \kappa$).

Theorem 1 follows from Theorem 2 by induction on ordinals $< \kappa^+$, $a_x(x \in U)$ being chosen to belong to $I$. See [8, Theorem 1].

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Theorem 2 for $\kappa = \omega$ is trivial—let $I = P(\omega) - U$. Thus from now on, $\kappa > \omega$. Theorem 2 follows from Lemma 1 and Lemma 1 clearly follows from Lemmas 2 and 3.

**Lemma 1.** If $U \in \mu \kappa$, then there is a $\kappa$-complete ideal $I \subseteq P(\kappa) - U$ which is dense in $U \mod (< \kappa)$.

To prove Theorem 2, let $I$ be as in Lemma 1 and set $x \in \bar{I}$ iff there are $x_0, x_1$ such that $x = x_0 \cup x_1$, $x_0 \in I$ and there is no $y \in I$ with $y \subseteq x_1$ and $|y| = \kappa$. Then $\bar{I}$ is as required.

For $f, g \in \kappa \kappa$ we shall write $f \sim g \pmod U$ or $f < g \pmod U$ if

\[
\{ \rho : f(\rho) = g(\rho) \} \subseteq U \quad \text{or} \quad \{ \rho : f(\rho) < g(\rho) \} \subseteq U \quad \text{respectively}.
\]

**Definition 2 [2].** $U \in \mu \kappa$ is regular if there is a family $X = \{ x_\xi : \xi \in \kappa \} \subseteq U$ such that for every infinite $S \subseteq \kappa$, $\bigcap \{ x_\xi : \xi \in S \} = 0$.

**Definition 3 [6].** $f \in \kappa \kappa$ is almost one-to-one if for each $\rho \in \kappa$, $|f^{-1}(\{ \rho \})| < \kappa$.

**Definition 4.** $f \in \kappa \kappa$ is bounded mod $U$ if for some $\alpha \in \kappa$, $\{ \rho \in \kappa : f(\rho) < \alpha \} \subseteq U$. Otherwise $f$ is unbounded mod $U$.

**Definition 5.** $U \in \mu \kappa$ is weakly selective if for every $f \in \kappa \kappa$ which is unbounded mod $U$ there is an almost one-to-one $g \in \kappa \kappa$ such that $f \sim g \pmod U$.

**Lemma 2.** If $U \in \mu \kappa$ is either regular or not weakly selective, then there is a $\kappa$-complete $I \subseteq P(\kappa) - U$ which is dense in $U \mod (< \kappa)$.

**Proof.** See [8, Theorems 5 and 11]. In outline, the proofs are as follows. Suppose first that $U$ is regular. Let $X = \{ x_\xi : \xi \in \kappa \}$ be as in Definition 2. Set

\[
I = \{ y \subseteq \kappa : \exists \eta < \kappa \forall \xi (\eta < \xi < \kappa \rightarrow |x_\xi \cap y| < \kappa) \}.
\]

Then $I$ is the desired ideal.

If $U$ is not weakly selective, fix $f \in \kappa \kappa$ unbounded mod $U$ and such that there is no almost one-to-one $g$ with $g \sim f \pmod U$. Set

\[
I = \{ y \subseteq \kappa : \forall \rho < \kappa (|y \cap f^{-1}(\{ \rho \})| < \kappa) \}.
\]

**Lemma 3.** If $U \in \mu \kappa$ is weakly selective and not regular, then there is a $\kappa$-complete ideal $I \subseteq P(\kappa) - U$ which is dense in $U \mod (< \kappa)$.  

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The next crucial lemma needed in the proof of Lemma 3 is due to A. Kanamori [6].

**Lemma 4 [6].** If \( U \in \mu \kappa \) is not regular then there is a least \((\text{mod } U)\) almost one-to-one function \( f \in \kappa^\kappa \).

**Proof in outline.** Suppose that Lemma 4 is false. One can then construct by induction almost one-to-one functions \( f_\alpha \in \kappa^\kappa \) (\( \alpha \in \kappa \)) such that for all \( \alpha < \beta < \kappa \), \( f_\beta < f_\alpha \) \((\text{mod } U)\), and in addition, if \( \beta \) is limit, then for all \( \alpha < \beta \) and all \( \rho \in \kappa \), \( f_\beta (\rho) \leq f_\alpha (\rho) \). We now define sets \( x_\alpha \in U \) \((\alpha < \kappa, \alpha \text{ successor})\) as follows. If \( \alpha = \gamma + n \) where \( \gamma < \kappa \), \( \gamma \) is limit and \( 1 \leq n \in \omega \), then

\[
x_\alpha = \{ \rho < \kappa : \forall m \in \omega (0 \leq m < n \rightarrow f_\alpha (\rho) < f_{\gamma + m} (\rho)) \}.
\]

It can be shown that \( \{ x_\alpha : \alpha < \kappa, \alpha \text{ successor} \} \) regularizes \( U \).

**Lemma 5.** If \( U \in \mu \kappa \) is weakly selective and not regular then there is a least \((\text{mod } U)\) \( f \in \kappa^\kappa \) unbounded \((\text{mod } U)\).

**Proof.** Immediate from Lemma 4.

**Lemma 6.** (Scott, see [7, Theorem 1.8]). Let \( U \in \mu \kappa \) and \( f \in \kappa^\kappa \) be as in Lemma 5. Set \( V = \{ x \subseteq \kappa : f^{-1} (x) \in U \} \). Then \( V \in \mu \kappa \) and the identity is a least \((\text{mod } V)\) function unbounded \((\text{mod } V)\). Moreover \( V \) extends the filter of closed unbounded subsets of \( \kappa \).

**Sketch of the proof of Lemma 3.** Let \( U, f \) and \( V \) be as in Lemma 6. Let \( J \) be the ideal of those \( x \subseteq \kappa \) such that \( \kappa - x \) contains a closed unbounded subset of \( \kappa \). By Lemma 6, \( J \subseteq \mathcal{P}(\kappa) - V \). It is well known that \( J \) is \( \kappa \)-complete and dense \((\text{mod } < \kappa)\). Set

\[
I = \{ y \subseteq \kappa : \exists x \in J (y \subseteq f^{-1} (x)) \}.
\]

**References**


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