THE NUMBER OF ZEROES OF AN ANALYTIC FUNCTION IN A CONE

BY CARLOS A. BERENSTEIN

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It is not possible to estimate the number of zeroes of an analytic function of several variables defined in a cone by reducing the problem to the 1-dimensional case via Crofton's formula or similar tools of Nevanlinna theory (see e.g. [4]). We propose to extend the classical result due to Pfluger and Levin [3] using a potential theory approach.

Let \( S^{m-1} \) be the unit sphere in the euclidean space \( \mathbb{R}^m \), \( D \) an open subset of \( S^{m-1} \), \( \partial D \) smooth and of bounded curvature. For \( 0 < r < \infty \) we set \( D_r = \{ tx \in D, 0 < t < r \} \). Denote by \( \rho_1 = \rho_1(D) \) the positive number such that \( \rho_1(\rho_1 + m - 2) \) is the first eigenvalue of the Laplace-Beltrami operator in \( D \) for the Dirichlet problem. Thus we have

**THEOREM.** Let \( u \) be a subharmonic function in \( D_\infty \), such that \( u \not\equiv -\infty, u(x) \leq A + B|x|^{\rho} \) for every \( x \in D_\infty \). If \( \rho > \rho_1, D' \) is an arbitrary open set, \( \bar{D}' \subseteq D \), then there exists a constant \( C = C(D', \rho) \) such that

\[
\lim_{r \to \infty} \frac{1}{r^{\rho - m + 2}} \int_{D'_r} \Delta u \leq CB.
\]

If we identify \( C^n \) with \( \mathbb{R}^{2n} \) and \( f \) is an analytic function in \( D \), then \( \log|f(z)|^2 \) is subharmonic and

\[
\sigma_D(r) = \frac{(n-1)!}{2\pi^n} \int_{D'_r} \Delta \log|f(z)|^2
\]

represents the euclidean area of the variety \( \{ z \in D_r : f(z) = 0 \} \). For \( n = 1 \), it is just the number of zeroes of \( f \) in \( D_r \); see [2]. Therefore we obtain the following

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COROLLARY. Let $f$ be a nonzero analytic function in $D_\infty$ satisfying $|f(z)| \leq A \exp(B|z|^\rho)$ for some $\rho > \rho_1$; then for any $D'$ open, $\overline{D'} \subseteq D$, we have

$$\lim_{r \to \infty} \sigma_D(r)r^{-\rho - m + 2} \leq CB.$$ 

The details of the proof and related results appear in [1], here we just present the bare bones of the proof of the theorem. First, we can show that

(i) the harmonic measure of $S_r = \{rx: x \in D\}$ at a fixed point $x_0$, behaves like $O(r^{-\rho_1})$,

(ii) if $G_r(x)$ is the Green function of $D_r$ with pole at $x_0$, $0 < \epsilon < 1$ fixed, then

$$G_r(x) \geq \text{const } r^{-\rho_1 - m + 2}, \quad r \to \infty$$

for $x \in D_{r'}$, $|x| > r_0$,

(iii) we can reduce the general case to the one in which $u \leq 0$ on $\partial D_\infty$.

Then we apply Green's formula, assuming $u(x_0) \neq -\infty$,

$$\int_{D_r} G_r(x)\Delta u = -u(x_0) + \int_{\partial D_r} u(x) \frac{\partial G_r}{\partial \nu}(x)$$

($\partial/\partial \nu$ derivative in the direction of the inner normal). By (i), (iii) and the assumption on $u$ we have

$$\int_{D_r} G_r(x)\Delta u = O(r^{\rho - \rho_1}) \quad \text{as } r \to \infty.$$ 

Applying (ii), the conclusion of the Theorem follows.

REFERENCES


