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DERIVATIVES OF ENTIRE FUNCTIONS AND A QUESTION OF PÓLYA

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The purpose of this note is to announce a partial solution to an old question of Pólya.

To state concisely his question and our results we introduce the following notation: For each integer $p \geqslant 0$, denote by V_{2p} the class of entire functions of the form $f(z) = \exp(-az^{2p+2})g(z)$ where $a \geqslant 0$ and $g(z)$ is a constant multiple of a real entire function of genus $\leqslant 2p + 1$ with only real zeros. Recall that a real entire function is one which assumes real values on the real axis. Now set $U_0 = V_0$, and for $p \geqslant 1$, set $U_{2p} = V_{2p} - V_{2p-2}$.[2]

The class U_0, often called the Laguerre-Pólya class, is of particular interest, for a classical theorem of Laguerre [4] and Pólya [6] asserts that $f \in U_0$ if and only if it can be uniformly approximated on discs about the origin by a sequence of polynomials with only real zeros. Consequently, if $f \in U_0$, then $f^{(n)} \in U_0$, $n = 1, 2, \cdots$; in particular, $f \in U_0$ implies $f^{(n)}$ has only real zeros $n = 1, 2, \cdots$.

In 1914, Pólya [7] asked whether the converse is true: *If an entire function f and all its derivatives have only real zeros, is $f \in U_0$?*

Pólya showed [7], [8] that if $f = Pe^Q$ where P and Q are polynomials, then, except for functions of the form ae^{bz} (a and b constants, b complex), the answer is yes. Moreover, in [8] he conjectured that, in the general case, the *only* exceptions are functions f of the form $f(z) = ae^{bz}$ or $f(z) = a(e^{icz} - e^{id})$ where a, b, c, and d are constants, c and d real, b complex. In [1] and [2], M. Ålander proved that the answer to Pólya's question is affirmative for all $f \in U_{2p}$ with $p \leqslant 2$ and in [3] purported to have extended this result to arbitrary p. However, in a famous survey article on zeros of successive derivatives [9], Pólya refers to Ålander's papers [1] and [2] but not to his more general result [3]. The first author of this announcement while a

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[2]This classification was introducted by Ålander [3].

graduate student under the direction of A. Edrei brought this curious omission to the latter's attention. In response to Edrei's subsequent query, Pólya replied in a letter that he was aware of Ålander's more general "proof" but was never convinced by it nor could he show that it was fallacious!

Ålander's "proof" involves a study of level curves of harmonic functions associated with functions in U_{2p}. Avoiding such geometric considerations, and using instead direct analytic arguments, we have succeeded in proving the following stronger version of Ålander's "theorem."

THEOREM 1. *Let $f \in U_{2p}$. If f' has only real zeros, then f'' has exactly $2p$ complex (i.e., nonreal) zeros.*

This result also partially affirms a long standing conjecture of Wiman [3, p. 6]: *If $f \in U_{2p}$, then f'' has at least $2p$ complex zeros.*

From Theorem 1 follows

COROLLARY 1. *Let f be a (constant multiple of a) real entire function of finite order. If f, f', and f'' have only real zeros, then $f \in U_0$.*

The case of real functions of infinite order has been studied by B. Ja. Levin and I. V. Ostrovskiĭ in [5]. Their results yield Corollary 1 for functions f of infinite order for which (roughly speaking) $M(r, f) = \sup_{|z|=r}|f(z)|$ grows asymptotically faster than $\exp(\exp r)$. Thus, for real entire functions, only those of infinite order and "moderate" growth remain to be studied to completely answer Pólya's question.

For arbitrary functions of finite order, we have verified Pólya's conjecture by our

THEOREM 2. *Let f be an entire function of finite order. If f, f', and f'' have only real zeros then either $f \in U_0$ or f has one of the forms $f(z) = ae^{bz}$, $f(z) = a(e^{icz} - e^{id})$ where a, b, c, and d are constants, b complex, c and d real.*

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