AN ENTROPY EQUIDISTRIBUTION PROPERTY FOR A MEASURABLE PARTITION UNDER THE ACTION OF AN AMENABLE GROUP

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Throughout this note let $G$ be an arbitrary discrete amenable group. Let $(\Omega, M, \lambda)$ be a probability space. Let $\mathcal{A}$ be the automorphism group of $(\Omega, M, \lambda)$. Let $T: G \to \mathcal{A}$ be a group homomorphism. We call $T$ an action of $G$ on $\Omega$. For each $g \in G$, let $T^g$ be the image of $g$ in $\mathcal{A}$ under $T$. Then $T^g$ is a measurable, measure-preserving, invertible map from $\Omega$ to itself.

If $Q$ is a partition of $\Omega$ and $\omega \in \Omega$, let $Q(\omega)$ be the element of $Q$ which contains $\omega$. If $E$ is a set let $|E|$ denote the cardinality of $E$.

Let $K$ be a subgroup of $G$. A net $\{A_\alpha\}$ of finite nonempty subsets of $K$ is said to satisfy property $P$ with respect to $K$ if $\lim\alpha |A_\alpha|^{-1} |gA_\alpha \cap A_\alpha| = 1, g \in K$. (Since $K$ is amenable, such a net $\{A_\alpha\}$ exists; see [3].)

Let $P$ be a measurable partition of $\Omega$ with finite entropy. If $E$ is a finite nonempty subset of $G$, let $h_p(E) \in L^1(\Omega)$ be defined as follows:

$$h_p(E)(\omega) = -\log \lambda \left\{ \bigvee_{g \in E} (T^g)^{-1} P \right\}(\omega), \quad \omega \in \Omega.$$  

The following generalization of the Shannon-McMillan theorem may be found in [4] and [8]: Let $G = Z^k$, where $Z$ is the group of integers and $k$ is a positive integer. For $n = 1, 2, \cdots$, let $A_n = \{(x_1, x_2, \cdots, x_k) \in Z^k: 0 \leq x_i \leq n, i = 1, 2, \cdots, k\}$. Then $\{ |A_n|^{-1} h_p(A_n) \}$ converges in $L^1(\Omega)$ as $n \to \infty$.

In [7] it is shown that if $G$ is the group of dyadic rationals modulo one, and if $A_n$ is the cyclic subgroup of $G$ generated by $2^{-n}$, then $\{ |A_n|^{-1} h_p(A_n) \}$ converges in $L^1(\Omega)$ as $n \to \infty$. The authors of [7] conjectured that this property generalizes to a general countable abelian group.


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It is the purpose of this note to announce the following theorem which generalizes these results, and settles the above conjecture. (The proofs of Theorems 1–4 will appear elsewhere.) Following [7], we call Theorem 1 the entropy equidistribution property of a measurable partition under the action of an amenable group.

**Theorem 1.** Let $K$ be a subgroup of the amenable group $G$. There exists a $K$-invariant function $h(P, T, K) \in L^1(\Omega)$ such that for every net $\{A_\alpha\}$ satisfying property $P$ with respect to $K$, $\lim_{\alpha} |A_\alpha|^{-1} h_P(A_\alpha) = h(P, T, K)$ in $L^1(\Omega)$.

The main tool used in proving Theorem 1 is the following generalized ergodic theorem which appears in [1]: If $K$ is a subgroup of $G$, $\{A_\alpha\}$ is a net satisfying property $P$ with respect to $K$, and $f \in L^1(\Omega)$, then $\{A_\alpha|^{-1} \sum_{g \in A_\alpha} f \cdot T^g\}$ has a limit in $L^1(\Omega)$ which is $K$-invariant.

Define $H(P, T, K) = \int h(P, T, K) \, d\lambda$. Define $C(K) = \{M \in M : \lambda[T^g(M) \Delta M] = 0, g \in K\}$.

**Theorem 2.** If $K_1$ and $K_2$ are subgroups of $G$ such that $K_1 \subseteq K_2$, then $H(K_2) \leq H(K_1)$. Equality holds if and only if $E[h(P, T, K_1)|C(K_2)] = h(P, T, K_2)$.

**Theorem 3.** If $K$ is a subgroup of $G$, there exists a countable subgroup $L$ of $K$ such that if $L'$ is any subgroup satisfying $L \subseteq L' \subseteq K$, then $h(P, T, L') = h(P, T, K)$.

**Theorem 4.** Let $K$ be a subgroup of $G$. Let $K$ be a family of subgroups of $K$ which is directed by inclusion ($\supset$), and whose union is $K$. Then $\lim_{L \in K} h(P, T, L) = h(P, T, K)$ in $L^1(\Omega)$, and $H(P, T, K) = \inf_{L \in K} H(P, T, L)$.

As an application of the foregoing results, we can define the entropy $H(T)$ of the action $T$ of the amenable group $G$ on $\Omega$ as follows: $H(T) = \sup_P H(P, T, G)$, where the supremum is over all measurable partitions $P$ of $\Omega$ with finite entropy. This definition extends that given in [2] for $G = \mathbb{Z}^k$. The entropy as we have defined it is an invariant under isomorphism. Conversely, it may be possible to generalize Ornstein’s results [6] and show that generalized Bernoulli schemes (see [5] for definition) with the same entropy are isomorphic. The entropy equidistribution property (Theorem 1 above) might serve as a basic tool for proving this.
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