A 4-MANIFOLD WHICH ADMITS NO SPINE

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1. This note is to present a new example which reveals the impossibility of embedding a 2-torus in a 4-manifold.

THEOREM 1. There exists a compact 4-dimensional PL manifold \( W^4 \) with boundary satisfying the following conditions: (i) \( W^4 \) is homotopically equivalent to the 2-torus \( T^2 = S^1 \times S^1 \), and (ii) no homotopy equivalence \( T^2 \rightarrow W^4 \) is homotopic to a PL embedding.

By a PL embedding is meant one which is not necessarily locally flat.

Theorem 1 is an application of the codimension two surgery theory developed in our previous papers [4], [5], [6]. The phenomena of "total spinelessness" in higher dimensions (with finite \( \pi_i \)'s) were found by Cappell and Shaneson [2] using another method of surgery [1].

A calculation in our proof leads to another consequence concerned with submanifolds in codimension two. Let \( K^{4n} \) denote a product \( CP^2 \times \cdots \times CP^2 \) of \( n \)-copies of the complex projective plane \( CP^2 \).

THEOREM 2. For each \( n \geq 0 \), there exists a locally flat embedding \( h^{(4n)} \) of \( K^{4n} \times S^1 \) into the interior of \( K^{4n} \times D^2 \times S^1 \), which is homotopic to the zero cross section \( K^{4n} \times \{0\} \times S^1 \), but is not locally flatly concordant to a splitted embedding.

A splitted embedding (with respect to a point \( * \) of \( S^1 \)) means a locally flat embedding \( f: K^{4n} \times S^1 \rightarrow K^{4n} \times D^2 \times S^1 \) such that (i) \( f \) is transverse regular to \( K^{4n} \times D^2 \times \{*\} \) so that the intersection \( M^{4n} = f(K^{4n} \times S^1) \cap K^{4n} \times D^2 \times \{*\} \) is a closed manifold, and (ii) the inclusion \( M^{4n} \rightarrow K^{4n} \times D^2 \times \{*\} \) is a homotopy equivalence.

Theorem 2 contrasts with Farrell and Hsiang's result [3] which may be

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2Their theory (with \( \Gamma \)-groups) and ours (with \( P \)-groups) are not the same but both admit a more general unifying algebraic treatment [7].
considered as the splitting theorem in codimension $\geq 3$.

2. Construction of $W^4$. Let $h : S^1 \to S^1 \times D^2$ be an embedding indicated in Figure 1. Essentially the same embedding $S^1 \to S^1 \times S^2$ was used by Mazur [8] to construct a contractible 4-manifold.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Mazur's embedding}
\end{figure}

Extend $h$ to a framed embedding $\tilde{h} : S^1 \times D^2 \to S^1 \times D^2$ in such a way that $\tilde{h}$ followed by the natural inclusion $S^1 \times D^2 \to S^3$ is isotopic to a trivial knot with a trivial framing. Our manifold $W^4$ is the mapping torus of the framed embedding $\tilde{h}$. More precisely, $W^4$ is obtained from a product $S^1 \times D^2 \times [0, 1]$ by identifying $(x, \xi) \times \{1\}$ with $\tilde{h}(x, \xi) \times \{0\}$ for each $(x, \xi) \in S^1 \times D^2$. Since $h$ is homotopic to the zero cross section $S^1 \times \{0\} \to S^1 \times D^2$, $W^4$ is homotopically equivalent to $T^2$.

Moreover, the embedding $h_{(4n)}$ in Theorem 2 is nothing other than $id_K \times h : K^{4n} \times S^1 \to K^{4n} \times S^1 \times D^2$, $h$ being Mazur's one.

3. Sketch of proof. We first give some generalities. Suppose a compact connected oriented PL $2n$-manifold $V^{2n+2}$ has the same simple homotopy type as an oriented Poincaré complex of formal dimension $2n \geq 6$. Let $\pi \to \pi'$ denote the associated (onto) homomorphism with $V^{2n+2}$ defined to be $\pi_1(V - L) \to \pi_1(V)$, where $L^{2n}$ is an exterior $n$-connected (i.e. taut) $2n$-sub-manifold of $V^{2n+2}$ [4]. The kernel of $\pi \to \pi'$ is generated by a (specified) central element $t$ represented by a fiber of the associated $S^1$-bundle with a 2-Disk bundle neighbourhood $N$ of $L^{2n}$.

A $(-1)^n$-Seifert form over $\pi \to \pi'$ is, by definition, a (not necessarily nonsingular) $(-1)^n\pi'$-Hermitian form defined over $Z\pi$ which becomes nonsingular over $Z\pi'$ (after tensored with $Z\pi'$).
Then the left $\mathbb{Z}^n$-module $\pi_{n+1}(V - L, N - L)$ is proved to carry a $(-1)^n$-Seifert form whose class in $P_{2n}(\pi \to \pi')$, the "Witt group" of $(-1)^n$-Seifert forms over $\pi \to \pi'$, does not depend on $L$. Denote the class by $\eta(V) \in P_{2n}(\pi \to \pi')$. Then $\eta(V) = 0$ if and only if $V$ admits a locally flat spine [6].

Now with the notations of §2, the product $W^4 \times \mathbb{C}P^2$ has the homotopy type of $T^2 \times \mathbb{C}P^2$. The associated homomorphism with it is $[Z \times Z \times Z \to Z \times Z] = (Z \to 1) \times Z \times Z$, and the obstruction element $\eta(W^4 \times \mathbb{C}P^2)$ is proven to be in the image of the injective homomorphism

$$j_*: P_6((Z \to 1) \times Z) \to P_6((Z \to 1) \times Z \times Z).$$

Let $\eta' = j_*^{-1}(\eta(W^4 \times \mathbb{C}P^2))$.

**Lemma 1.** The element $\eta'$ of $P_6((Z \to 1) \times Z)$ is represented by a $(-1)$-Seifert form $(G, \lambda, \mu)$ given by: $G = \Lambda x_1 \oplus \Lambda x_2$, $\lambda(x_1, x_2) = -s^{-1}$, $\mu(x_1) = s - 1$, $\mu(x_2) = -1$, where $\Lambda = Z[t, t^{-1}, s, s^{-1}]$, $t$ (or $s$) denoting the positive generator of the first (or the second) $Z$ of $(Z \to 1) \times Z$.

**Remark.** The matrix $(\lambda(x_i, x_j))$ of the $(-1)$-Seifert form of Lemma 1 is

$$\begin{pmatrix} (s-1) - (s^{-1}-1)t, & -s^{-1} \\ st, & -1 + t \end{pmatrix},$$

the determinant of which coincides (up to units) with the Alexander polynomial of Mazur's link (Figure 1) calculated by the method of Torres and Fox [9].

**Lemma 2.** $\eta'$ is not in the image of

$$i_*: P_6(Z \to 1) \to P_6((Z \to 1) \times Z).$$

The proof of Theorem 1 goes as follows. Suppose that there were a spine $T^2_0 \subset W^4$. $T^2_0$ may be assumed to be locally flat except at one point. The product $T^2_0 \times \mathbb{C}P^2$ is a spine of $W^4 \times \mathbb{C}P^2$ whose singularity is of the type (knot cone) $\times \mathbb{C}P^2$. Since $\pi_1(\{pt\} \times \mathbb{C}P^2) = \{1\}$, this singularity is replaced by a knot cone singularity over a knotted 5-sphere in a 7-sphere [4], [6, §6.4]. This implies that the $\eta(W^4 \times \mathbb{C}P^2)$ is in the image of $j_* \circ i_*$, since $P_6(Z \to 1)$ is isomorphic to the $(7, 5)$-knot cobordism group [6]. However, this contradicts Lemma 2.

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This notation slightly differs from the original one [6].
REMARK. If we start the construction with the embedding indicated in Figure 2, we will obtain $W^4$ which admits a locally flat spine.

![Figure 2. False embedding](image)

REFERENCES


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