THE q-REGULARITY OF LATTICE POINT PATHS IN \( \mathbb{R}^n \)

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1. Introduction. Given any set \( X \) and a cardinal number \( q \), then, following Rado \([4]\), a collection \( S \) of sets is called \( q \)-regular in \( X \) if, whenever \( X \) is partitioned into \( q \) parts, then at least one part contains as a subset a member of \( S \). More generally, by requiring the partitions of \( X \) to belong to a given preassigned family \( F \), we obtain the notion of \( q \)-regularity in \( X \) relative to \( F \).

Letting \( N \) denote the positive integers, given \( q \in N \), it is convenient to regard a partition of a set \( X \) into \( q \) parts as a function \( f: X \to Z_q \), where \( Z_q \) denotes the ring of integers modulo \( q \). The partition \( P(f) = \{ f^{-1}(\overline{m}) : m \in N \} \) of \( X \) is said to be represented by \( f \), where \( \overline{m} \) denotes the residue class in \( Z_q \) containing \( m \in N \). Given \( f: \mathbb{R} \to Z_q, g: \mathbb{R} \to Z_q \), then, as in \([2]\), we obtain a partition \( f \oplus g: \mathbb{R} \times \mathbb{R} \to Z_q \) by the formula \( (f \oplus g)(x, y) = f(x) + g(y) \), where the sum on the right takes place in \( Z_q \).

If \( A \subseteq \mathbb{R}^n \) is a subset of euclidean \( n \)-space \( \mathbb{R}^n \), let \( F^o(A) \) denote the family consisting of those partitions of \( A \) which are representable by functions \( f_1 \oplus \cdots \oplus f_n|A: A \to Z_q \), where \( f_i: \mathbb{R} \to Z_q, i = 1, \cdots, n \), and where \( g|A \) denotes the restriction of the function \( g \) to \( A \).

A (linear) lattice point path in \( \mathbb{R}^n \) shall mean the intersection of a connected subset of a straight line in \( \mathbb{R}^n \) with the lattice points \( \mathbb{Z}^n \subseteq \mathbb{R}^n \), where \( Z \) denotes the set of integers. Adding a maximal element \( \infty \) to \( R \), and given any \( j \in N^* = N \cup \{\infty\} \), let \( L_j \) denote the collection of lattice point paths obtainable from lines which have a set of integer direction numbers bounded in absolute value by \( j \), and let \( S_{j,k} \subseteq L_j \) denote the subcollection of \( L_j \) consisting of those paths which contain \( k \) points, \( k \in N^* \).

Given any \( A \subseteq \mathbb{R}^n, j \in N^* \), and \( q \in N \), we then define

\[
\rho_{j,q}(A) = \sup \{ k \in N : S_{j,k} \text{ is } q\text{-regular in } A \},
\]

\[
\rho^o_{j,q}(A) = \sup \{ k \in N : S_{j,k} \text{ is } q\text{-regular in } A \text{ relative to } F^o(A) \},
\]

where we set \( \rho_{j,q}(A) = \rho^o_{j,q}(A) = 0 \) if \( A \cap \mathbb{Z}^n = \emptyset \).
Note that the functions $\rho_{j,q}, \rho_{j,q}^\bullet$ are monotone, and that $\rho_{j,q}(A) \leq \rho_{j,q}^\bullet(A)$ for all $A \subseteq \mathbb{R}^n$. The case $j = 1, q = 2$ is of special interest, so that we then suppress the subscripts, writing $\rho = \rho_{1,2}, \rho^\bullet = \rho_{1,2}^\bullet$. For example, one of our main results is the formula $\rho^\bullet(\mathbb{Z}^n) = n, n \in \mathbb{N}$, where we conjecture that this formula also holds when $\rho^\bullet$ is replaced by $\rho$ (it does hold for $\rho$ when $n \leq 3$). Also, letting $\mathcal{C}^n(m)$ denote an $n$-dimensional hypercube of lattice points having $m$ points on a side, $\mathcal{C}^n(m) = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n: 1 \leq x_i \leq m, i = 1, \ldots, n\}$, note that $\rho(\mathcal{C}^n(m)) \leq m$, where the equality $\rho(\mathcal{C}^n(m)) = m$ can be interpreted to imply that $n$-dimensional Tic-Tac-Toe cannot be played to a tie in $\mathcal{C}^n(m)$ (where a winning set in $\mathcal{C}^n(m)$ consists of $m$ points in a straight line). The following proposition has a simple verification.

**Proposition 1.** $\rho^\bullet(\mathcal{C}^n(n)) = n, n \in \mathbb{N}$.

2. **Statement of main results.** The following three theorems represent our main results on $\rho, \rho^\bullet, \text{ and } \rho_{\infty,2}^\bullet$.

**Theorem 1.** $\rho^\bullet(\mathbb{Z}^n) = n, n \in \mathbb{N}$.

**Theorem 2.** $\rho(\mathcal{C}^n(n)) \leq n - 1, n \geq 4$.

**Theorem 3.** $\rho_{\infty,2}^\bullet(\mathbb{Z}^n) \leq 2n - 1, n \in \mathbb{N}$.

**Remarks.** 1. Theorem 2 is surprising in view of the contrasting fact that $\rho(\mathcal{C}^n(n)) = n, n \leq 3$ (compare also with Proposition 1). Hales and Jewett have shown [2, Theorem 5] that the winning sets in $\mathcal{C}^n(n + 1)$ are not 2-regular in $\mathcal{C}^n(n + 1), n \in \mathbb{N}$, i.e., in our terminology, $\rho(\mathcal{C}^n(n + 1)) \leq n$. They actually show (again in our terminology) that $\rho^\bullet(\mathcal{C}^n(n + 1)) \leq n$, although it turns out that the partitions they use cannot be extended to partitions of $\mathbb{Z}^n$ satisfying the requirements of Theorem 1. Note that the result $\rho(\mathcal{C}^n(n + 1)) \leq n$ also follows immediately from Theorem 1, while Theorem 2 improves this latter result in the dimensions $n \geq 4$. Even in case the winning sets in $\mathcal{C}^n(m)$ are not 2-regular in $\mathcal{C}^n(m)$, it still might not be possible for the second player to force a tie. For results on when the second player can force a tie, see [1] and [2].

2. To obtain a function dependent upon $\rho_{j,q}$, but which, unlike $\rho_{j,q}$, is invariant under affine isomorphisms of $\mathbb{R}^n$, we define, for $A \subseteq \mathbb{R}^n$,

$$\lambda_{j,q} = \text{sup} \{\rho_{j,q}(f(A)): f: \mathbb{R}^n \to \mathbb{R}^n \text{ is an affine map}\},$$

with $\lambda_{j,q}^\bullet$ defined similarly using $\rho_{j,q}^\bullet$ in place of $\rho_{j,q}$. Setting $\lambda = \lambda_{1,2}, \lambda^\bullet = \lambda_{1,2}^\bullet$, we see from Proposition 1 and Theorem 1 that $\lambda^\bullet$ distinguishes in a
natural geometric-combinatorial way amongst the various euclidean spaces. Indeed, we have the following corollary, which we conjecture also holds for $\lambda$.

**Corollary 1.** $\lambda^\theta(U) = n$, whenever $U$ is a nonempty open set in $R^n$, $n \in N$.

3. **Description of the partitions used in our main results.** Given $m \in N$, let $\tau_m : Z \rightarrow Z$, $\phi_m : Z \rightarrow Z$ be defined by $\tau_m(x) = x + m$, $\phi_m(x) = [x/m]$, where $[y]$ denotes the greatest integer $\leq y$. Theorems 1, 2, and 3 depend on a remarkable sequence $\{g_n\}$ of functions from $Z$ into $Z_2$ defined by the formulas

I. $g_1(x) = x$ $(x \in Z)$,

II. $g_{2m} = g_1 \circ \phi_{2m}$ $(m \in N)$,

III. $g_{n-1} = g_n + g_n \circ \tau_1$ $(n \geq 2)$.

These are overdefinitions, but turn out to be consistent. Theorem 1 is verified using the function $g_1 \oplus \cdots \oplus g_n : Z^n \rightarrow Z_2$, while Theorem 3 is verified using $g_n \oplus \cdots \oplus g_{2n-1} : Z^n \rightarrow Z_2$. Theorem 2 is verified by the restriction, to a suitable translate of $C^*(n)$, of $f \circ g_1$, when $n = 4$, and of $f \circ g_1 \oplus g_4 \oplus \cdots \oplus g_{n-1}$, when $n \geq 5$, where $f : Z^3 \rightarrow Z_2$ is suitably defined. The proofs that the above functions do the job depend on a rather involved analysis of the subgroup of $(Z_2)^Z$ generated by $g_1, \cdots, g_n$. This analysis, together with additional details and results, is contained in [3].

**REFERENCES**


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