The $\mu$-invariant $\mu(M)$ of an oriented $\mathbb{Z}_2$-homology 3-sphere $M$ is defined by Hirzebruch in [8], using Rohlin's Theorem [13], to be the mod 16 reduction of the signature of a framed manifold $W$ with $M = \partial W$. In this paper we give a formula for $\mu(M)$ by studying $M$ as a branched dihedral covering space of $S^3$. Hilden [7] and Montesinos [9] have independently shown that every closed orientable 3-manifold is actually a 3-fold (dihedral) covering space of $S^3$ branched along a knot.\(^1\) Also see [1], [6] and [12].

Let $\alpha$ be a smooth or piecewise linear oriented knot $S^1 \subset S^3$. Let $V \subset S^3$ with $\partial V = \alpha$ be a Seifert surface for $\alpha$. The Seifert form $L = L_V$ is the bilinear form of linking numbers of circles in $V$, with respect to a fixed orientation of $S^3$, and $L'$ is given as $L'(x, y) = L(y, x)$. Let $p$ be an odd integer. A knot $\beta$ will be called a mod $p$ characteristic knot (in $V$) of $\alpha$ if there is an embedding of $S^1$ in $V$, with nontrivial homology class $[\beta] \in H_1(V)$, so that the composite $S^1 \subset V \subset S^3$ is $\beta$; and if $L(x, \beta) + L(\beta, x) \equiv 0 \pmod{p}$ for all $x$ in the image of $\pi_1^V \times \mathbb{Z}_p$.

A mod $p$ characteristic knot $\beta$ for $\alpha$ determines a homomorphism $\rho$ of $\pi_1(S^3 - \alpha)$ onto the dihedral group $\mathbb{Z}_2 \times \omega \mathbb{Z}_p$ of order $2p$. The map $\rho$ is characterized by the requirements that its composition with $\mathbb{Z}_2 \times \omega \mathbb{Z}_p \to \mathbb{Z}_2$ be nontrivial and that, for $x$ in the image of $\pi_1 V$, $\rho(x) \in \mathbb{Z}_p \subset \mathbb{Z}_2 \times \omega \mathbb{Z}_p$ is the mod $p$ reduction of $L(x, \beta)$. Hence $\beta$ determines a $p$-fold dihedral branched covering $M_{\alpha, \beta}$ of $S^3$, branched along $\alpha$. It can be shown that every dihedral representation for $\alpha$ and associated branched cover of $S^3$ are determined by a characteristic knot for $\alpha$ in $V$. Further, dihedral representations can easily be classified in terms of equivalence classes of characteristic knots. By abuse of notation, we write $M_\alpha$ for $M_{\alpha, \beta}$; as “most” knots have at most one (up to conjugacy) dihedral representation of order $2p$, this notation is usually strictly justified.

\(^1\)In fact one can go directly from a Heegard splitting to a description of any orientable 3-manifold as a dihedral branched covering space.
Let \( \alpha_0, \cdots, \alpha_{(p-1)/2} \) be the disjoint oriented circles in \( M_\alpha \) that lie over \( \alpha \), with \( \alpha_0 \) of branching index 1 and \( \alpha_i \) of index 2, \( 1 \leq i \leq (p - 1)/2 \). Orient \( M_\alpha \) so that the covering projection has positive degree, and let \( v_{ij} \) denote the linking number of \( \alpha_i \) with \( \alpha_j \), \( i \neq j \). If \( M_\alpha \) is a homology sphere, then \( v_{ij} \) is an integer, but for a \( Z_2 \)-homology sphere \( v_{ij} \) will in general be a fraction with odd denominator. Let

\[
v_{ij} = -\left( \sum_{j=1, j \neq i}^{(p-1)/2} v_{ij} + v_{i0}/2 \right), \quad v_0 = -2 \sum_{i=1}^{(p-1)/2} v_{i0}.
\]

Let \( J \) be the matrix \((v_{ij})_{1 \leq i, j \leq (p-1)/2}\). K. Perko introduced \( v_0 \) and has computed \( v_{ij} \) and \( v_0 \) for many knots.

Let \( \Sigma_\beta \) be the \( p \)-fold cyclic branched cover of \( S^3 \), branched along \( \beta \), and oriented so that the covering projection \( \tau \) has positive degree. For \( p \) a prime-power \( \Sigma_\beta \) is a rational homology sphere [5]. Let \( T \) be a covering translation corresponding to a meridian about \( \beta \). Then

\[
\tau^{-1} V = \overline{V} \cup_\beta T(\overline{V}) \cup_\beta \cdots \cup_\beta T^{p-1}(\overline{V})
\]

where \( \tau|\overline{V} : \overline{V} \to V \) is a homeomorphism, and where \( \overline{\beta} = \tau^{-1}(\beta) \). Let \( z_1, \cdots, z_r \) be elements in the image of \( H_1(V - \beta) \) in \( H_1(V) \) which, together with \([\beta]\), form a basis (over \( \mathbb{Q} \)) for this image. Let \( A_i \) be the matrix whose \((j, k)\)th entry is the linking number in \( \Sigma_\beta \) of \((\tau|\overline{V})^{-1}_{*} z_j \) and \( T^i((\tau|\overline{V})^{-1}_{*} z_k) \), \( 1 \leq i \leq p - 1 \). Let \( A \) have the \((j, k)\)th entry \( L_V(z_j, z_k) \). Let \( R = ((R_{ij})) \), \( 1 \leq i, j \leq (p-1)/2 \) be the matrix of blocks where, with subscripts modulo \( p \),

\[
R_{ij} = A_{i-j} + A_{j-i} - A_{i+j} - A_{-i-j}, \quad i \neq j,
\]

and

\[
R_{ii} = A + A' - 2(A_1 + \cdots + A_{p-1}) - A_{2i} - A_{-2i}.
\]

For any knot \( \eta \) in a \( Z_2 \)-homology sphere, let \( \Delta_\eta(t) \) denote its Alexander polynomial.

Let \( \hat{\alpha} \) be any knot obtained from \( \alpha_1, \cdots, \alpha_{(p-1)/2} \) by connected sum using \((p-3)/2 \) paths joining them. (Such paths may be described by lifting suitable paths from \( \alpha \) to itself in \( S^3 \).)

For any fraction \( p/q \), \( p \) and \( q \) odd, let \( \varphi(p/q) = 0 \) if \( p/q \equiv \pm 1 \) (mod 8) and \( \varphi(p/q) = 8 \) if \( p/q \equiv \pm 3 \) (mod 8).

\[\text{Footnote:} \quad \text{When } p \text{ is not a prime-power and } \Sigma_\beta \text{ not a rational homology sphere, the definition of the matrix } R \text{ is slightly more complicated.}\]
If $N$ is a Hermitian matrix, let $\sigma(N)$ denote its signature. Let $\psi$ be a primitive $p$th root of unity. Let $B$ be a Seifert matrix for $\beta$.

**Theorem.** Suppose that the branched $p$-fold dihedral covering space $M_\alpha$ of $S^3$ is a $\mathbb{Z}_2$-homology sphere. Assume that $v_{i,0} \equiv 2 \pmod{4}$ for $1 \leq i \leq (p - 1)/2$. Then the following holds modulo 16:

$$
\mu(M_\alpha) = \sum_{i=1}^{p-1} \sigma(B + B' - B\psi^i - B'\psi^{-i}) + \left(\frac{p-1}{2}\right) \varphi(\Delta_\alpha(-1))
$$

$$
+ \varphi(p) L_Y([\beta], [\beta]) + \varphi(\Delta_\alpha(-1) \Delta_\alpha(1)) - (v_0/4) + \sigma(J) - \sigma(R).
$$

The terms on the right of this formula are readily calculable. Note that some of the terms vary with the choice of characteristic knot $\beta$. (However, for suitable dihedral covers of ribbon knots these terms contribute zero. This gives a simple obstruction to a knot being a ribbon knot.) For a homology sphere $M_\alpha$, the terms on the right are integers. For “bushel baskets” [6] of knots $\alpha$, $\pi_1(M_\alpha) = 0$. Some of the terms vary with the choice of characteristic knot $\beta$. For a homology sphere $M_\alpha$, the terms on the right are integers. For “bushel baskets” [6] of knots $\alpha$, $\pi_1(M_\alpha) = 0$.

The condition on $v_{i,0}$ seems to be satisfied in all known cases for $M_\alpha$ a $\mathbb{Z}_2$-homology sphere [10], [11]. It implies that $\det J \neq 0$, which suffices for the theorem. Therefore it seems reasonable to conjecture at least that $\det J \neq 0$ if $M_\alpha$ is a $\mathbb{Z}_2$-homology sphere. For $p = 3$, we can show that $v_{1,0} = 2 \pmod{4}$. (See [2] and [10] for 2-bridge knots.) Hence the Theorem applies in this case. By [7] and [9] the case $p = 3$ of our formula applies to every $\mathbb{Z}_2$-homology sphere.

**References**


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For $M$ a homotopy 3-sphere, $\mu(M) = 0$ if $M \times S^1 \times S^1$ is P.L. homeomorphic to $S^3 \times S^1 \times S^1$ and $\mu(M) \neq 0$ if $M \times S^1 \times S^1$ is P.L. homeomorphic to the exotic manifold described in [14] which is homotopy equivalent but not P.L. homeomorphic to $S^3 \times S^1 \times S^1$. 

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