VOLTERRA-STIELTJES INTEGRAL EQUATIONS WITH LINEAR CONSTRAINTS AND DISCONTINUOUS SOLUTIONS

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X and Y denote Banach spaces; we consider systems of the form

(K) \[ y(t) - y(t_0) + \int_{t_0}^{t} dK(t, a) \cdot y(a) = f(t) - f(t_0), \]

(F) \[ F[y] = c, \]

where \( y, f \in G([a, b], X) \) (the space of regulated functions \( g: [a, b] \to X \), i.e., \( g \) has only discontinuities of the first kind); \( K \in G^{uo} \) (see §2) and \( F \in L[G([a, b], X), Y] \) (linear constraint). (K) includes linear Volterra integral equations, linear delay differential equations, differential equations \( y' + A'y = f' \), with the meaning that we have

(L) \[ y(t) - y(s) + \int_{s}^{t} dA(a) \cdot y(a) = f(t) - f(s) \quad \text{for all } s, t \in [a, b]. \]

In §2 we give the existence of the resolvent for (K) and in §3 for (L); in §4 we find the Green function for the system (K), (F). The results of §1 are used in the proofs. All results of this announcement may be extended to open intervals and \( Y \) a separated sequentially complete locally convex TVS.

The proofs will appear in [H.3].

1. A division of \([a, b]\) is a finite sequence \( d: t_0 = a < t_1 < \cdots < t_n = b \). We write \(|d| = n\) and \( \Delta d = \sup_{1 \leq i \leq n} |t_i - t_{i-1}| \). The set \( D \) of all divisions of \([a, b]\) is ordered by refinement and \( \lim_{d \in D} x_d \) denotes the limit according to the associated net. For \( \alpha: [a, b] \to L(X, Y) \) and \( f: [a, b] \to X \) we define the usual Riemann-Stieltjes operator integral

\[ \int_{a}^{b} d\alpha(t) \cdot f(t) = \lim_{\Delta d \to 0} \sum_{i=1}^{d} [\alpha(t_i) - \alpha(t_{i-1})] \cdot f(\xi_i) \]

where \( \xi_i \in [t_{i-1}, t_i] \) (see [G], [H.1], [D]), and the interior integral

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\[
\int_a^b d\alpha(t) \cdot f(t) = \lim_{d \to 0} \sum_{i=1}^{d} [\alpha(t_i) - \alpha(t_{i-1})] \cdot f(\xi_i^*)
\]

where \(\xi_i^* \in ]t_{i-1}, t_i[\) (see [K], [H, p. 96]), when these limits exist. The existence of the first integral implies the existence of the second one and reciprocally, if \(\alpha\) and \(f\) are bounded with no common discontinuity. We define

\[
SV[\alpha] = SV_{[a,b]}[\alpha] = \sup_{d \in D} SV_d[\alpha]
\]

where

\[
SV_d[\alpha] = \sup \left\{ \left\| \sum_{i=1}^{d} [\alpha(t_i) - \alpha(t_{i-1})] \cdot x_i \right\| \left| x_i \in X, \|x_i\| \leq 1 \right. \}
\]

If \(SV[\alpha] < \infty\) we say that \(\alpha\) is of bounded semivariation and we write \(\alpha \in SV([a, b], L(X, Y))\); if we have further \(\alpha(a) = 0\) we write \(\alpha \in SV_0([a, b], L(X, Y))\).

For \(u: [a, b] \to L(X, Y)\) we define \(S[u] = \sup_{d \in D} S_d[u]\), where

\[
S_d[u] = \sup \left\{ \left\| \sum_{i=0}^{d} u(t_i) \cdot x_i \right\| \left| x_i \in X, \|x_i\| \leq 1 \right. \}
\]

and we write \(u \in S([a, b], L(X, Y))\) if \(S[u] < \infty\). For \(f \in G([a, b], X)\) we define \(f_-(t) = f(t-)\) if \(a < t < b\) and \(f(a-) = 0\); we write \(f \in G_-([a, b], X)\) if \(f_-=f\) and \(f \in c_0([a, b], X)\) if \(f_- = 0\).

**Theorem 1.** The mapping

\[
(\alpha, u) \in SV_0([a, b], L(X, Y)) \times S([a, b], L(X, Y)) \mapsto F = F_\alpha + F_u \in L[G([a, b], X), Y]
\]

defines a bicontinuous isomorphism of the first Banach space onto the second, where for \(f \in G([a, b], X)\) we define

\[
F_\alpha[f] = \int_a^b d\alpha(t) \cdot f(t) \quad \text{and} \quad F_u[f] = \sum_{a \leq t \leq b} u(t) \cdot [f(t) - f(t-)].
\]

We have \(\|F_\alpha\| = SV[\alpha], \alpha(t) \cdot x = F[X_{\alpha t}x]\) and \(u(t) \cdot x = F[X_{\{t\}}x]\).

For \(X = Y = R\) this theorem is due to Kaltenborn [K].

**Theorem 2.** Given \(\alpha \in SV([c, d], L(X, Y)), h: [c, d] \times [a, b] \to L(X)\) which is a regulated function in the first variable and uniformly of bounded semivariation in the second variable (i.e., \(h^t \in SV([a, b], L(X))\) for every
$t \in [c, d]$ and $\sup_{c \leq r \leq d} SV[h^t] < \infty$, where $h^t(s) = h(t, s)$ and $g \in G([a, b], X)$ we have $\overline{h} \in SV([a, b], L(X, Y))$, and $\overline{g} \in G([c, d], X)$, where

$$\overline{h}(s) = \int_c^d d\alpha(t) \cdot h(t, s)$$
and

$$\overline{g}(t) = \int_a^b d_s h(t, s) \cdot g(s),$$

and

\begin{equation}
\int_a^b \left[ \int_c^d d\alpha(t) \cdot h(t, s) \right] g(s) = \int_c^d d\alpha(t) \left[ \int_a^b d_s h(t, s) \cdot g(s) \right].
\end{equation}

If $[c, d] = [a, b]$ and $g$ is continuous we have the formula of Dirichlet

\begin{equation}
\int_a^b \left[ \int_c^d d\alpha(t) \cdot h(t, s) \right] d\alpha(s) = \int_a^b d\alpha(t) \cdot \left[ \int_t^b h(t, s) \cdot d\alpha(s) \right].
\end{equation}

If $[c, d] = [a, b]$, $\alpha \in A_\alpha$ (see §3) and $h \in G^{uo}$ (see §2) we have (2).

**Remark.** (1) generalizes a theorem of Bray proved for $X = Y = R$ [B].

2. For $U: [a, b] \times [a, b] \to L(X)$ we consider the following properties

\begin{enumerate}[\emph{(SV\textsuperscript{o})}]
\item $\lim_{\delta \to 0} SV_{[s-\delta, s+\delta]} [U^t] = 0$ for all $s, t \in [a, b]$,
\item $\lim_{\delta \to 0} \sup_{s, t} SV_{[s-\delta, s+\delta]} [U^t] = 0$.
\end{enumerate}

We write $U \in G^{uo}$ if $U$ is bounded, regulated as a function of the first variable and satisfies (SV\textsuperscript{o}). $G^{uo}$ is a Banach space when endowed with the norm $\|U\| = \|U\| + \sup_{a \leq t \leq b} SV[U^t]$.

**Theorem 3.** Given $K \in G^{uo}$ we have:

I. There is one and only one element $R \in G^{uo}$ (i.e. $R \in G^{uo}$ and $R(t, t) \equiv I_X$), the resolvent of (K), such that

$$R(t, s) = I_X - \int_t^s d_\alpha K(t, \alpha) \cdot R(\alpha, s) \quad \text{for all } s, t \in [a, b].$$

II. For every $f \in G([a, b], X)$ the equation (K) with $y(t_0) = x$ has one and only one solution $y \in G([a, b], X)$ given by

$$y(t) = R(t, t_0)x + \int_{t_0}^t R(t, \sigma) \, df(\sigma)$$

and $y$ depends continuously on $f, x$ and $K$.

III. If $K \in G^{uo}$ (i.e. $K \in G^{uo}$ and $K(t, t) \equiv 0$) we have

$$R(t, s) = I_X + \int_t^s R(t, \alpha) \cdot d_\alpha K(\alpha, s) \quad \text{for all } s, t \in [a, b].$$
IV. The mapping $K \in C^0_u \mapsto R \in C^0_t$ is a bicontinuous (nonlinear) bijection from the first space onto the second.

REMARK. Theorem 3 remains true if we replace $G^u$ by its subspace $E^u$ of continuous functions, by its subspace $E^c$ of functions $U$ that satisfy

$$\lim_{t \to t_1} SV[U^t - U^{t_1}] = 0$$

for every $t_1 \in [a, b]$,

by the corresponding spaces of functions of bounded variation, etc.

3. We now particularize Theorem 3 to (L). We fix a point $\bar{\sigma} \in [a, b]$; given $A : [a, b] \to L(X)$ we write $A \in \mathcal{A}_{\bar{\sigma}}$ if $A(\bar{\sigma}) = 0$ and if $A$ satisfies $(SV^o)$. $(SV^o_u), (SV^o_0), (SV^o_c)$ denote the analogous for the first variable of the properties $(SV^u), (SV^o), (SV^c)$ in the second variable. We say that $R : [a, b] \times [a, b] \to L(X)$ is harmonic, and we write $R \in \mathcal{H}$, if $R$ satisfies $(SV^u), (SV^c), (SV^o_u), (SV^o_c)$ and

$$R(t, t) \equiv I_X, \quad R(t, \sigma) \circ R(\sigma, s) = R(t, s) \text{ for all } s, \sigma, t \in [a, b].$$

$\mathcal{H}^c$ denotes $\mathcal{H}$ with the topology induced by $E^u$; analogously we define $\mathcal{H}^c$. The next theorem extends Theorems 3.2 and 3.3 of [M].

THEOREM 4. A. Given $A \in \mathcal{A}_{\bar{\sigma}}$ we have:

I. There is one and only one $R \in \mathcal{H}$, the resolvent of $A$, such that

$$R(t, s) = R(\tau, s) - \int_\tau^t dA(\tau) \circ R(\tau, s) \text{ for all } s, \tau, t \in [a, b].$$

II. For every $f \in G([a, b], X)$ the equation (L) with $y(s) = x$ has one and only one solution $y \in G([a, b], X)$ given by

$$y(t) = R(t, s)x + \int_s^t R(t, \sigma)df(\sigma)$$

and $y$ depends continuously on $f, x$ and $A$.

III. $A(t) = \int_t^\sigma d_\sigma R(\sigma, s) \circ R(s, o)$ for all $s \in [a, b]$ and

$$R(t, s) = R(t, o) + \int_o^t R(t, \tau) \circ dA(\tau) \text{ for all } s, o, t \in [a, b].$$

B. If $R : [a, b] \times [a, b] \to L(X)$ satisfies $(o)$ and $(SV_0)$ then $R \in \mathcal{H}$ and $R$ is the resolvent of $A$ given in III.

C. On $\mathcal{H}$ the topologies of $\mathcal{H}^c$ and $\mathcal{H}^c$ coincide and the mapping $A \in \mathcal{A}_{\bar{\sigma}} \mapsto R \in \mathcal{H}$ is a bicontinuous (nonlinear) bijection from the first space onto the second.
4. We now consider the problem (K), (F) with \( K \in \mathbb{E}^w \); we write \( K[y] = f \) for (K) and define \( Y_0 = F[K^{-1}(0)] \). Let \( \alpha \) be associated to \( F \) by Theorem 1; for \( s \in [a, b] \) we define \( J(s) = \int_a^b da(t) \circ R(t, s) \).

**Theorem 5.** The following properties are equivalent:

(i) \( y \equiv 0 \) is the only solution of the system \( K[y] = 0, \ F[y] = 0 \).

(ii) \( J(t_0) : X \to Y_0 \) is a continuous bijection.

From now on we suppose that the equivalent properties (i), (ii) are satisfied and that

\[
\left\{ \int_a^b da(t) \cdot f(t) \mid f \in G([a, b], X) \right\} = Y_0.
\]

We define

\[
\tilde{J}(t) = R(t, t_0) \circ J(t_0)^{-1} : Y_0 \to X
\]

and

\[
G(t, s) = \tilde{J}(t) \circ \int_a^s da(\tau) \circ R(\tau, s) - Y(s-t_0)\tilde{J}(t) \circ J(s) + [Y(s-t_0) - Y(s-t)] R(t, s).
\]

**Theorem 6.** A. The system \( K[y] = g, \ F[y] = c \) has a solution \( y \in C([a, b], X) \) iff \( (g, c) \in C([a, b], X) \times Y_0 \); then this solution is

\[
y(t) = \tilde{J}(t)c + \int_a^b G(t, s) \, dg(s).
\]

B. The system \( K[y] = f, \ F[y] = c \) has a solution \( y \in G([a, b], X) \) iff \( c - F(f) \in Y_0 \); then this solution is given by

\[
y(t) = f(t) + \tilde{J}(t)[c - F(f)] - \int_a^b G(t, s) \, ds \left[ \int_t^s da K(s, a) \right] f(a).
\]

**Theorem 7.** The Green function \( G : [a, b] \times [a, b] \to L(X) \) has the following properties:

- \((G_0)\) \( F[G_s] = 0 \) for every \( s \in [a, b] \), where \( G_s(t) = G(t, s) \).
- \((G_1)\) \( G_s(t) - G_s(t_0) + \int_{t_0}^t da K(t, a) \circ G_s(a) = [Y(s-t) + Y(s-t_0)] I_X \).
- \((G_2)\) \( G(t,s) + \int_a^t da \tilde{G}(t, a) \circ K(a, s) = \tilde{J}(t) \circ \alpha(s) \) where

\[
\tilde{G}(t, o) = G(t, o) + Y(o-t) R(t, o) + Y(o-t_0)[\tilde{J}(t) \circ J(o) - R(t, o)].
\]

- \((G_3)\) For every \( s \in [a, b] \), \( G_s \) is continuous for \( t \neq s \).
- \((G_4)\) \( G \) is uniformly of bounded semivariation in the second variable;

\( G(t, b) \equiv 0; \ G(t, a) = 0 \) for \( a < t < b \), \( G(a, a) = -I_X \).
REFERENCES


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