WHY ANY UNITARY PRINCIPAL SERIES REPRESENTATION OF $SL_n$ OVER A $p$-ADIC FIELD DECOMPOSES SIMPLY

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Recently A. Knapp [3] announced that every unitary principal series representation of a semisimple Lie group decomposes simply; that is, no two distinct irreducible components of a given unitary principal series representation are equivalent. In proving this result Knapp analyzed in detail the structure of the spaces of intertwining operators for principal series representations. His analysis used a detailed description, due to Harish-Chandra, of the Fourier transform on semisimple Lie groups. In this note we want to prove the analogue of Knapp's result for $SL_n$ over a $p$-adic field. Our proof will be conceptually quite different from Knapp's. Although the case we consider is admittedly quite special, there is hope that this approach to the problem will generalize.

Let $F$ be a nonarchimedean local field and let $G = GL_n(F)$. Let $D$ be the diagonal subgroup of $G$ and $U$ the upper unipotent matrices. Let $B = D \cdot U$ be the group of all nonsingular upper triangular matrices and write $\delta$ for the modular function of $B$. (If $d_b$ is a left Haar measure for $B$, then $\delta(b)d_b$ is a right Haar measure.) Let $K$ be a maximal compact subgroup of $G$ and recall that $G = KB$.

Let $\chi$ be any (unitary) character of $D$ and regard it as a character of $B$. The induced representation $\pi_\chi = \text{Ind}_B^G(\chi S^{1/2})$ is called the (unitary) principal series representation attached to $\chi$. To describe the unitary representation $\pi_\chi$ explicitly we let $H_\chi$ denote the Hilbert space of all complex-valued measurable functions $h$ on $G$ such that $h(gb) = \chi^{-1}(b)\delta^{-1/2}(b)h(g)$ ($g \in G, b \in B$) and such that $\int_K |h(k)|^2dk < \infty$. Then $\pi_\chi$ is just left translation in $H_\chi$: $(\pi_\chi(x)h)(g) = h(x^{-1}g)$ ($h \in H_\chi; g, x \in G$).

We use the subscript "1" to denote the subgroup of $G_1 = SL_n(F)$ ob-


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tained by intersecting a given group with $G_1$. Observe that, for any $\chi$, $\pi_\chi|G_1 = \text{Ind}_{B_1}^{G_1} (\chi \delta^{1/2})$, where $\chi_1 = \chi|B_1$ (or $\chi|D_1$). In words, the restriction of a unitary principal series representation from $G$ to $G_1$ is a unitary principal series representation of $G_1$.

The purpose of this note is to prove

**Theorem.** $\pi_{\chi_1}$ decomposes into finitely many inequivalent irreducible unitary representations.

We will give two proofs for this theorem. The first connects better with the existing literature and appears to be more generally applicable. The second relates to the germ expansion for the character of $\pi_{\chi_1}$; perhaps this proof gives a clearer picture of why the decomposition is simple.

**Proof (1).** The group $B$ acts on the linear characters of the unipotent group $U$ and there is a unique open orbit $U$ for this action. Fix an element $\eta \in U$. Let $V_\chi \subseteq H_\chi$ denote the space of locally constant functions in $H_\chi$, i.e. the space of admissible vectors. Then $V_\chi$ is a dense $G$-submodule of $H_\chi$. It is known that there is, up to a constant, a unique nonzero linear form $\Phi$ on $V_\chi$ such that $\Phi(\pi_\chi(u) h) = \eta(u) \Phi(h)$ for every $h \in V_\chi$ and $u \in U$ [4]. We call $\Phi$ the Whittaker form on $V_\chi$.

The representation $\pi_{\chi_1} = \pi_\chi|G_1$ decomposes into finitely many irreducible unitary subrepresentations and this decomposition respects $V_\chi$, i.e. $V_\chi = V_1 \oplus \ldots \oplus V_r$, where each $V_i$ ($i = 1, \ldots, r$) is an irreducible admissible unitary $G_1$-module. Let $\alpha_i : V_\chi \rightarrow V_i$ be the projection of $V_\chi$ on $V_i$ corresponding to this decomposition. Since $U \subset G_1$, we see that the adjoint $\alpha_i^*$ of $\alpha_i$ maps $\Phi$ to a Whittaker form $\alpha_i^*(\Phi)$ on $V_i$. It follows that $\alpha_i^*(\Phi) \neq 0$ for only one index $i_0$, $1 \leq i_0 \leq r$. Therefore, for $j \neq i_0$, $V_j$ is not $G_1$-equivalent to $V_{i_0}$, i.e. up to equivalence the $G_1$-module $V_{i_0}$ occurs only once in $V_\chi$. On the other hand, since $G$ acts irreducibly on $V_\chi$ [2], the diagonal group $D$ permutes the $G_1$-modules $V_i$ transitively. Thus, each $V_i$ has some Whittaker form defined on it and, by the same argument as before, each $V_i$ occurs, up to equivalence, only once in the $G_1$-module $V_\chi$. Thus, $\pi_{\chi_1}$ decomposes simply.

**Proof (2).** For the second proof we will assume that $n$ is not divisible by the characteristic of $F$. In this case the index, $[G : G_1 \cdot F^x] < \infty$ and the number of unipotent conjugacy classes in $G_1$ is finite. Let $\xi$ denote the character of $\pi_\chi$ and let $\xi_i$ denote the character of the $G_1 \cdot F^x$-module $V_i$, so $\xi|G_1 \cdot F^x = \sum \xi_i$. Harish-Chandra has recently observed that the arguments
of [1] apply to all characters; thus, in a neighborhood of $1 \in G_1 \cdot F^x$, each $\xi_i$ has an expansion of the form $\xi_i = \Sigma a_{ij} \hat{\psi}_j$, where the $\hat{\psi}_j$ are certain distributions attached to the unipotent conjugacy classes in $G_1$ (indexed by the letter $f$) and the $a_{ij}$ are complex constants. Summing, we have $\xi = \Sigma (\Sigma a_{ij}) \hat{\psi}_j$. It follows from [5] that, for unipotent conjugacy classes of maximum dimension (i.e. "regular" classes), the corresponding constants $a_{ij}$ must be either 0 or 1. Moreover, the existence of the Whittaker form shows that, for at least one $\xi_i$ and for some regular class, the coefficient is 1. Conjugating by $D$, we again see, since $G$ acts irreducibly on $V_\chi$, that, for every $\xi_i$, there is a regular class for which the coefficient is 1. Since the sum of these coefficients over $i$ is, for any fixed regular unipotent class, at most 1, it follows that the sets of regular unipotent classes associated in this way to different $\xi_i$’s must be disjoint. Again it follows that the $V_i$’s are nonisomorphic $G_1$-modules, i.e. their characters are all different.

REFERENCES

5. F. Rodier, Modèles de Whittaker et caractères de représentations (preprint).