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WHY ANY UNITARY PRINCIPAL SERIES REPRESENTATION OF SL_n OVER A p-ADIC FIELD DECOMPOSES SIMPLY

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Recently A. Knapp [3] announced that every unitary principal series representation of a semisimple Lie group decomposes simply; that is, no two distinct irreducible components of a given unitary principal series representation are equivalent. In proving this result Knapp analyzed in detail the structure of the spaces of intertwining operators for principal series representations. His analysis used a detailed description, due to Harish-Chandra, of the Fourier transform on semisimple Lie groups. In this note we want to prove the analogue of Knapp's result for SL_n over a p-adic field. Our proof will be conceptually quite different from Knapp's. Although the case we consider is admittedly quite special, there is hope that this approach to the problem will generalize.

Let F be a nonarchimedean local field and let $G = GL_n(F)$. Let D be the diagonal subgroup of G and U the upper unipotent matrices. Let $B = D \cdot U$ be the group of all nonsingular upper triangular matrices and write δ for the modular function of B. (If d_lb is a left Haar measure for B, then $\delta(b)d_lb$ is a right Haar measure.) Let K be a maximal compact subgroup of G and recall that $G = KB$.

Let χ be any (unitary) character of D and regard it as a character of B. The induced representation $\pi_\chi = \mathrm{Ind}_B^G(\chi\delta^{1/2})$ is called the (unitary) principal series representation attached to χ. To describe the unitary representation π_χ explicitly we let H_χ denote the Hilbert space of all complex-valued measurable functions h on G such that $h(gb) = \chi^{-1}(b)\delta^{-1/2}(b)h(g)$ $(g \in G,\ b \in B)$ and such that $\int_K |h(k)|^2 dk < \infty$. Then π_χ is just left translation in H_χ: $(\pi_\chi(x)h)(g) = h(x^{-1}g)$ $(h \in H_\chi;\ g, x \in G)$.

We use the subscript "1" to denote the subgroup of $G_1 = SL_n(F)$ ob-

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tained by intersecting a given group with G_1. Observe that, for any χ, $\pi_\chi|G_1 = \mathrm{Ind}_{B_1}^{G_1}(\chi_1\delta^{1/2})$, where $\chi_1 = \chi|B_1$ (or $\chi|D_1$). In words, the restriction of a unitary principal series representation from G to G_1 is a unitary principal series representation of G_1.

The purpose of this note is to prove

THEOREM. *π_{χ_1} decomposes into finitely many inequivalent irreducible unitary representations.*

We will give two proofs for this theorem. The first connects better with the existing literature and appears to be more generally applicable. The second relates to the germ expansion for the character of π_{χ_1}; perhaps this proof gives a clearer picture of why the decomposition is simple.

PROOF (1). The group B acts on the linear characters of the unipotent group U and there is a unique open orbit $\mathcal{U}$ for this action. Fix an element $\eta \in \mathcal{U}$. Let $V_\chi \subset \mathcal{H}_\chi$ denote the space of locally constant functions in $\mathcal{H}_\chi$, i.e. the space of admissible vectors. Then V_χ is a dense G-submodule of $\mathcal{H}_\chi$. It is known that there is, up to a constant, a unique nonzero linear form Φ on V_χ such that $\Phi(\pi_\chi(u)h) = \eta(u)\Phi(h)$ for every $h \in V_\chi$ and $u \in U$ [4]. We call Φ the Whittaker η-form on V_χ.

The representation $\pi_{\chi_1} = \pi_\chi|G_1$ decomposes into finitely many irreducible unitary subrepresentations and this decomposition respects V_χ, i.e. $V_\chi = V_1 \oplus \ldots \oplus V_r$, where each V_i $(i = 1, \ldots, r)$ is an irreducible admissible unitary G_1-module. Let α_i: $V_\chi \longrightarrow V_i$ be the projection of V_χ on V_i corresponding to this decomposition. Since $U \subset G_1$, we see that the adjoint α_i^* of α_i maps Φ to a Whittaker η-form $\alpha_i^*(\Phi)$ on V_i. It follows that $\alpha_i^*(\Phi) \neq 0$ for only one index i_0, $1 \leqslant i_0 \leqslant r$. Therefore, for $j \neq i_0$, V_j is not G_1-equivalent to V_{i_0}; i.e. up to equivalence the G_1-module V_{i_0} occurs only once in V_χ. On the other hand, since G acts irreducibly on V_χ [2], the diagonal group D permutes the G_1-modules V_i transitively. Thus, each V_i has some Whittaker form defined on it and, by the same argument as before, each V_i occurs, up to equivalence, only once in the G_1-module V_χ. Thus, π_{χ_1} decomposes simply.

PROOF (2). For the second proof we will assume that n is not divisible by the characteristic of F. In this case the index, $[G\colon\ G_1 \cdot F^x] < \infty$ and the number of unipotent conjugacy classes in G_1 is finite. Let ξ denote the character of π_χ and let ξ_i denote the character of the $G_1 \cdot F^x$-module V_i, so $\xi|G_1 \cdot F^x = \Sigma\, \xi_i$. Harish-Chandra has recently observed that the arguments of [1] apply to all characters; thus, in a neighborhood of $1 \in G_1 \cdot F^x$, each ξ_i has an expansion of the form $\xi_i = \Sigma_j a_{ij} \hat{O}_j$, where the $\hat{O}_j$ are certain distributions attached to the unipotent conjugacy classes in G_1 (indexed by the letter j) and the a_{ij} are complex constants. Summing, we have $\xi = \Sigma_j(\Sigma_i a_{ij})\hat{O}_j$. It follows from [5] that, for unipotent conjugacy classes of maximum dimension (i.e. "regular" classes), the corresponding constants a_{ij} must be either 0 or 1. Moreover, the existence of the Whittaker form shows that, for at least one ξ_i and for some regular class, the coefficient is 1. Conjugating by D, we again see, since G acts irreducibly on V_χ, that, for every ξ_i, there is a regular class for which the coefficient is 1. Since the sum of these coefficients over i is, for any fixed regular unipotent class, at most 1, it follows that the sets of regular unipotent classes associated in this way to different ξ_i's must be disjoint. Again it follows that the V_i's are nonisomorphic G_1-modules, i.e. their characters are all different.

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