WHEN IS A MANIFOLD A LEAF OF SOME FOLIATION?

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Given a connected smooth open manifold $L$, does there exist a compact manifold $M$ and a $C^r$ codimension $q$ foliation of $M$ with a leaf diffeomorphic to $L$? Here $1 \leq r < \infty$. Most of our results are for $q = 1$, but note that if the answer is yes for $q$ then it is yes for any $q' > q$. Theorem 1 gives four conditions on $L$ any one of which is sufficient, and the Corollary provides interesting examples where $L$ is a surface. We have found no necessary condition in general, but Theorem 2 gives a strong necessary condition on the ends of $L$ in order that $L$ be a codimension one leaf each of whose ends has only one asymptote. Details and proofs will appear elsewhere.

**THEOREM 1.** $L$ is diffeomorphic to a leaf of a $C^r$ codimension $q$ foliation of some compact manifold if any one of the following conditions is satisfied ($q = 1$ except possibly in condition 1.4).

1.1. $L$ is diffeomorphic to the interior of a compact manifold-with-boundary ($r = \infty$ and $L$ will be a proper leaf).

1.2. $L = L_1 \# L_2$ where $L_1$ and $L_2$ are proper leaves of $C^r$ codimension one foliations of compact orientable manifolds.

1.3. $L = L_1 - X$ where $L_1$ is a leaf of a $C^r$ codimension one foliation of a compact manifold with a closed transversal which intersects $L_1$ in $X$.

1.4. $L$ is a regular covering space of a compact manifold with covering group which has a $C^r$ action on a connected compact $q$-manifold with a free orbit. (If the orbit is discrete, the leaf $L$ will be proper.)

Recall (see e.g. [2]) that an end $e$ of a connected manifold is determined by a sequence $U_1 \supset U_2 \supset \ldots$ of unbounded components of the complements of compact sets such that $\bigcap_{i=1}^{\infty} \overline{U}_i = \emptyset$. Another such sequence $V_1 \supset V_2 \supset \ldots$ determines the same end if every $U_i$ contains some $V_j$. Each $U_i$ is called a neighborhood of $e$. Define $e$ to be boundable if it has a closed neighborhood of the form $B \times [0, \infty)$ where $B$ is a connected compact manifold.

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Corollary. Every orientable 2-manifold with a finite number of ends is a proper leaf of a $C^r$ foliation of a compact 3-manifold, where $r = 1$ or $r = \infty$ depending on whether the number of non-boundable ends is odd or even, respectively.

If $e$ is an end of a leaf $L$ of a foliation of a manifold $M$, define the asymptote set $A_e$ of $e$ to be $\bigcap_{i=1}^{\infty} Cl(U_i)$, where $e$ is determined by neighborhoods $U_1 \supset U_2 \supset \ldots$ in $L$ and $Cl(U_i)$ denotes the closure of $U_i$ in $M$. Then $A_e$ is a well-defined closed union of leaves and is connected if $M$ is compact. Define a leaf $L$ to be nice if $A_e$ is a single leaf for every end $e$ of $L$. Note that a nice leaf is proper and that $A_e$ is compact if $M$ is compact. Finally, say that an end $e$ of a manifold $L$ is an infinite repetition if some closed neighborhood in $L$ of $e$ is of the form $W \cup F W \cup F \ldots$ where $W$ is a connected compact manifold-with-boundary, $\text{Bd} W$ has two components $\text{Bd}_+ W$ and $\text{Bd}_- W$, and $f$: $\text{Bd}_+ W \rightarrow \text{Bd}_- W$ is a diffeomorphism.

Theorem 2. If $L$ is a nice leaf of a $C^1$ codimension one foliation of a compact manifold then $L$ has only a finite number of ends and each one is an infinite repetition.

The proof uses the following two theorems, of which the first is a generalization of Reeb's first stability theorem in [3] and the second is proved using the framed surgery method of [1].

Theorem 3. Let $M$ be a (not necessarily compact) manifold-with-(possibly empty) boundary with a codimension $q$ foliation transverse to $\text{Bd} M$. Let $A$ be a compact leaf and let $D$ be a $q$-disk transverse to the foliation and cutting $A$ in exactly one point $x_0$. Suppose there exists a point $x$ in $D$ such that each element of the holonomy group of $A$ has a representative local diffeomorphism of $D$ whose domain contains $x$ and which leaves $x$ fixed. If $x$ is sufficiently closed to $x_0$ then the leaf through $x$ is diffeomorphic to $A$.

Theorem 4. If $h$: $\Pi_1 A \rightarrow \mathbb{Z}$ is a surjection, where $A$ is a connected compact manifold, then there exists a smooth map $g$: $A \rightarrow S^1$ such that $h = g_*$: $\Pi_1 A \rightarrow \Pi_1 S^1 = \mathbb{Z}$ and for some regular value $v$ in $S^1$, the manifold $g^{-1}(v)$ is connected and does not separate $A$.

References
