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Uniform distribution of sequences, by L. Kuipers and H. Niederreiter, Wiley, New York, 1974, xiv+390 pp., $24.50

The theory of uniform distribution started with Hermann Weyl's celebrated paper of 1916 titled *Ueber die Gleichverteilung von Zahlen mod Eins* [13]. In its initial stage the theory was deeply rooted in diophantine approximations. Later the subject became a meeting ground for number theory, probability theory, functional analysis and topological algebra. The vast literature on uniform distribution is therefore widely spread. The existing surveys, for example [6], [1], [7], give only a partial introduction to the theory. It is a very valuable enrichment of the mathematical literature that a book has been published which is at the same time an easily accessible introduction to the subject and an almost complete account of it.

Writing a book for both beginners and researchers in a field is an almost impossible task. The authors show that it was not impossible in this case. Firstly, the basic concepts and ideas of the theory are mostly elementary. Secondly, by proving only the main results, by inserting references and additional results in rather lengthy notes at the end of each section and by adding many exercises of various sorts to each section, the authors succeed in describing the underlying principles of the theory in such a way that readers neither get lost in generalizations nor are drowned in technicalities. Thirdly, the finish of the book is excellent; the text is well got-up and at the end there is an extensive bibliography and further a list of symbols and abbreviations, an author index and a subject index.

An indication of the contents might explain the subject and the scope of the book. Chapter 1 deals with the qualitative aspects of uniform distribution modulo one (u.d. mod 1). A sequence $(x_n)_{n=1}^{\infty}$ of real numbers is u.d. mod 1 if and only if for every continuous function $f: [0, 1] \to \boldsymbol{R}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)\,dx,$$

where $\{x\}=x-[x]$ denotes the fractional part of x. This implies that $(x_n)_{n=1}^{\infty}$ is u.d. mod 1 if and only if the proportion of the numbers x_n in any subinterval of [0, 1] is asymptotically equal to the length of the interval. A very useful criterion for u.d. mod 1 was given by Weyl, namely

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\exp(2\pi i h x_n)=0 \qquad \text{for all } h\in \boldsymbol{Z},\quad h\neq 0.$$

It follows, for example, that $(\theta n^k)_{n=1}^{\infty}$ for $\theta\notin \boldsymbol{Q}$, $k\in\boldsymbol{Z}$, $k>0$ and for $\theta\in\boldsymbol{R}$, $\theta\neq 0$, $k\in\boldsymbol{R}\backslash\boldsymbol{Z}$, $k>0$ and $(\theta\log^\tau n)_{n=1}^{\infty}$ for $\theta\neq 0$ and $\tau\in\boldsymbol{R}$, $\tau>1$ are u.d. mod 1, but that $(\theta \log^\tau n)_{n=1}^{\infty}$ is not u.d. mod 1 for $\tau\leqslant 1$. An example of a metric theorem is that the sequence $(x^n)_{n=1}^{\infty}$ is u.d. mod 1 for almost all $x>1$, i.e. apart from a set of Lebesgue measure 0. The further part of the chapter deals with several generalizations of the concept of u.d. mod 1.

Chapter 2 concerns the quantitative aspects of u.d. mod 1. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers and denote by $A([\alpha,\beta);N)$ the number among $x_1,\cdots,x_N$ with $\alpha\leqslant\{x_j\}<\beta$. We define the discrepancy D_N by

$$D_N=\sup_{0\leqslant\alpha<\beta\leqslant 1}\left|\frac{A([\alpha,\beta);N)}{N}-(\beta-\alpha)\right|.$$

Hence, (x_n) is u.d. mod 1 if and only if $\lim_{N\to\infty} D_N=0$. Recently W. M. Schmidt [9] improved upon previous estimates by proving the existence of an absolute constant $c>0$ such that for every sequence (x_n) one has $ND_N>c\log N$ for infinitely many N. On the other hand, Lerch [8] showed already in 1904 that for every algebraic irrational α there exists a constant $c=c(\alpha)>0$ such that $ND_N<c\log N$ for all N. The following result due to Koksma indicates the connection with the theory of numerical integration: Let f be a function on [0, 1] of bounded variation $V(f)$, and suppose that the N points $x_1,\cdots,x_N$ in [0, 1] have discrepancy D_N. Then

$$\left|\frac{1}{N}\sum_{n=1}^{N}f(x_n)-\int_0^1 f(t)\,dt\right|\leqq V(f)D_N.$$

In the third chapter the theory of uniform distribution in an arbitrary compact Hausdorff space with countable base is developed. Let X be such a space and let μ be a nonnegative regular normed Borel measure in X. Then a sequence (x_n) of elements in X is called μ-u.d. in X if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(x_n)=\int_X f\,d\mu \quad \text{for all } f\in\mathscr{R}(X),$$

where $\mathscr{R}(X)$ is the set of all real-valued continuous Borel-measurable functions on X. Several results proved for the classical case of u.d. mod 1 can be stated for this general case. For example the following Weyl criterion holds: In X there exists a countable convergence-determining class of functions in $\mathscr{R}(X)$ with respect to μ.

Results based on the algebraic structure of the reals cannot have an analogue in the theory of u.d. in X. To generalize this part of the classical theory, a theory of u.d. in compact topological groups is developed in Chapter 4. A sequence (x_n) in a compact group G is called u.d. in G if (x_n) is u.d. in G with respect to the Haar measure on G. A simple and attractive result is that if the sequence (a^n) is dense in G, then it is u.d. in G. The last part of the chapter is devoted to uniform distribution in locally compact groups.

The last and shortest chapter deals with the distribution of sequences in special domains, such as the ring of rational integers, rings of p-adic integers, and polynomial rings over finite fields. Here we restrict ourselves to sequences of integers. Let (a_n) be a sequence of rational integers and define $A(j, m; N)$ as the number of terms among $a_1, a_2, \cdots, a_N$ that satisfy the congruence $a_n \equiv j \pmod{m}$. Then (a_n) is u.d. mod m in case

$$(*) \qquad \lim_{N\to\infty} \frac{A(j, m; N)}{N} = \frac{1}{m} \quad \text{for } j = 1, \ldots, m.$$

Further, (a_n) is said to be u.d. in $\boldsymbol{Z}$ if $(*)$ holds for every integer $m \geqslant 2$. In this case the Weyl criterion for u.d. mod m reads

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \exp\left(\frac{2\pi i h a_n}{m}\right) = 0 \quad \text{for } h = 1, 2, \ldots, m-1.$$

There are several connections between u.d. in $\boldsymbol{Z}$ and u.d. mod 1. As a consequence the following sequences are u.d. in $\boldsymbol{Z}$: $([\theta n^k])_{n=1}^{\infty}$ for $\theta \notin \boldsymbol{Q}$, $k \in \boldsymbol{Z}$, $k>0$ and $([\theta \log^\tau n])$ for $\theta \neq 0$, $\tau > 1$.

It is evident that the book cannot cover the whole theory and the authors made a very good choice from the existing literature. However, one important aspect of uniform distribution is neglected, namely its applicability to other fields. The investigation of u.d. mod 1 of $(\theta n)_{n=1}^{\infty}$ has its origin in the theory of secular perturbations in astronomy and the theory of statistical mechanics. See for example Weyl **[12]**. At the same time Hardy and Littlewood **[3]** used the u.d. mod 1 of $(\theta n^k)_{n=1}^{\infty}$ to prove that $\zeta(1+it) = o(\log t)$. These applications led Weyl to the definition of uniform distribution and the development of the theory as mentioned in the beginning of this review **[11]**, **[13]**. A few years later Hardy and Littlewood **[4]** applied it to count the number of lattice points in a right-angled triangle. For a multidimensional generalization of this result, see Spencer **[10]**. Franel **[2]** has given an interesting equivalence between the Riemann hypothesis and some discrepancy inequality of the Farey points. See also Huxley, **[5]**. For the numerous applications to numerical integration I refer to the proceedings of a recent conference **[14]**. Many more applications have been given. Applications and open problems which would have important applications are hardly mentioned in the book, although most references can be found in the bibliography. I would have appreciated a survey of the applications with the pertinent references.

We are indebted to Professors Kuipers and Niederreiter for the manner in which they have written this book covering material which for a great part was the preserve of experts. The book will become a standard work and it will have a substantial impact on future research on uniform distribution.

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