BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 4, July 1975

FINITELY EMBEDDED MODULES OVER NOETHERIAN RINGS

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Communicated by Barbara L. Osofsky, March 3, 1975

A right R-module is *finitely embedded* if it has finitely generated essential socle (see, e.g., Vamos [4]). In [1] Jategaonkar settled the Jacobson conjecture for left and right fully bounded noetherian rings by showing that every finitely generated finitely embedded module is artinian. It is an open question whether or not this property holds for an arbitrary left and right noetherian ring (though it is well known that right noetherian is not enough). We prove here that if R is left and right noetherian and M is a projective finitely generated finitely embedded module over R, then M is artinian. This result can be extended to cover the case where M is an arbitrary finitely embedded submodule of a finitely generated free R-module. The proof of this and related results will appear elsewhere. Even the case $M = R$ seems to be new and in this case we can obtain the more general

THEOREM. *If R is a left and right noetherian ring and if the right socle of R is either left or right essential, then R is artinian.*

We note that there exist right noetherian rings with right essential right socle which are not right artinian (see, e.g., [2]).

All rings have identity and modules are unitary.

$l_A(X)$ and $r_A(X)$ denote, respectively, the left and right annihilators of X in the ring A.

We require the following result of T. H. Lenagan [3] which is adapted to suit our purpose in the form of

LEMMA 1. *Let S, R be rings and let R be right noetherian. Let M be an $(S\text{-}R)$-bimodule such that ${}_SM$ has finite length and M_R is finitely generated. Then $R/r(M)$ is an artinian ring and M_R has finite length.*

We also need the following lemma whose proof is straightforward.

LEMMA 2. *Let A be a subring of B and suppose that A is right*

AMS (MOS) subject classifications (1970). Primary 16A46.

noetherian and B_A *is finitely generated. Let e be an idempotent in B and* K_A *an A-essential submodule of* $(eB)_A$. *Then* $l_{eBe}(K)$ *is nilpotent.*

THEOREM 3 . *Let R be a left and right noetherian ring and let T denote the full* $(n \times n)$ *matrix ring over R. If e is an idempotent in T and* $(eT)_R$ *is finitely embedded, then* $(eT)_R$ *is artinian.*

PROOF. Let K denote the socle of $(eT)_R$. Now T_R and $(eT)_R$ are noetherian and eT is a left eTe right R-bimodule. Since TeT is left T-finitely generated it follows that eT is left eTe-finitely generated.

If I_R is a simple submodule of $(eT)_R$ and if $y \in T$, then either $eyI = 0$ or eyI is a simple R-submodule of eT. This shows that $eTeK$ is a subset of K and so K is an $(eTe\text{-}R)$-bisubmodule of eT. Since K_R has finite length, the left-right reverse of Lemma 1 gives that the ring $eTe/l_{eTe}(K)$ is artinian. By Lemma 2 eTe is artinian. Now eT has finite length as a left eTe-module, so by Lemma 1 again, $(eT)_R$ is artinian.

THEOREM 4. *Let R be a left and right noetherian ring and let P be a finitely generated projective right R-module. If P is finitely embedded then P is artinian.*

PROOF. Suppose that P has n generators. The direct sum P^n of n copies of P is R-isomorphic to an idempotently generated right ideal of the $(n \times n)$ matrix ring T over R. Now P^n is finitely embedded and therefore is artinian by Theorem 3. It follows that P is artinian.

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