EXTENSION THEOREMS FOR REDUCTIVE GROUP ACTIONS ON COMPACT KAHLER MANIFOLDS

BY ANDREW J. SOMMESE

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Let $G$ be a connected complex reductive Lie group. Noting [3], [7], [8] that $G$ has the structure of a linear algebraic group, let $G$ be any projective manifold in which $G$ is Zariski open and which induces the above algebraic structure on $G$. The purpose of the present note is to announce

**Proposition I.** Let $G$ be as above and act holomorphically on a compact Kaehler manifold $X$. Assume that the Lie algebra of holomorphic vector fields on $X$ generated by $G$ is annihilated by every holomorphic one form. Let $\Phi: Y \to X$ be a holomorphic map where $Y$ is a normal reduced analytic space. Consider the equivariant map $\Phi': G \times Y \to X$; $\Phi'$ extends meromorphically (in the sense of Remmert) to $\overline{G} \times Y$.

**Remarks.** The condition on vector fields annihilated by one forms is automatically satisfied if (cf. [12]—[14]) $H^1(X, \mathcal{O}) = 0$, or $G$ is semisimple, or if every generator of the solvable radical of $G$ has a fixed point, or if $G$ is a linear algebraic group acting algebraically on a projective $X$. Taking $Y$ to be a point, one gets the orbits of $G$ to be Zariski open in their closures which are analytic sets. A simple corollary is the classical result that there is only one structure of a linear algebraic group on $G$ (cf. [7]), and in fact any reductive connected subgroup of an algebraic group over $\mathbb{C}$ is an algebraic subgroup.

As a further application of the techniques used, a new proof of an improved form of a fixed point theorem (cf. [12], [13], [14]) of the author is given:

**Proposition II.** Let $S$ be a connected solvable Lie group acting holomorphically on a compact Kaehler manifold $X$. The following are equivalent:

(a) $S$ has a fixed point on $X$.

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(b) $S$ leaves a compact set in a fibre of the Albanese map invariant.

(c) $S$ has a fixed point within any compact set $K$ on $X$ that $S$ leaves invariant.

(d) The Lie algebra of vector-fields that $S$ generates on $X$ is annihilated by every holomorphic one form on $X$.

**Remark.** The assertion (c) where $K$ is any compact set is new; the method of proof allows one to relax the compactness of $X$ and show if, in addition, $H^1(X, O_X) = 0$, then (c) is true.

The following is the fundamental observation on which everything rests.

**Lemma.** Let $X$ be a compact Kähler manifold and $\rho: C^* \rightarrow \text{Aut}(X)$ a holomorphic $C^*$ action that has at least one fixed point. Let $A: C^* \rightarrow X$ be a holomorphic equivariant map onto an orbit: then $A$ extends to a homomorphic equivariant map $\tilde{A}$ of $CP^1$ to $X$.

**Proof.** Assume without loss of generality that $A(C^*)$ is not a point. Let $\mu$ be a Kähler metric on $X$ and $\omega$ the associated Kähler form. Assume that $\mu$ has been averaged with respect to the circle subgroup $S^1 \subseteq C^*$. Let $\chi$ be the holomorphic vector-field on $X$ associated to $\rho: C^* \rightarrow \text{Aut}(X)$.

Because of equivariance, the Jacobian, $dA$, of $A$, maps some constant multiple of $z(\partial/\partial z)$ onto the restriction of the vector-field $\chi$ to $A(C^*)$. Without loss of generality this constant is assumed to be one.

Let $A^*\mu = a(r) \, dz \otimes d\overline{z}$ where $a(r)$ is positive and depends only on $r$ due to the $S^1$ averaging of $\mu$. $A^*\omega = (i/2) a(r) \, dz \wedge d\overline{z}$.

$$
\mu(\chi, \chi) = \mu \left( dA \left( z \frac{\partial}{\partial z} \right), dA \left( z \frac{\partial}{\partial z} \right) \right) = A^*\mu \left( z \frac{\partial}{\partial z}, z \frac{\partial}{\partial z} \right)
$$

$$
= a(r)|z|^2 \leq M < \infty
$$

where $\sup_X \mu(\chi, \chi) = M < \infty$.

Now by Lichnerowicz [5] there exists a $C^\infty$ function $\phi$ on $X$ such that $\overline{\partial}\phi = \omega(\chi)$. Pulling back and, without confusion, letting $\phi$ stand for $A^*\phi = \phi(A(z))$, one has

$$
i \frac{1}{2} z a(r) \, d\overline{z} = \frac{\partial \phi}{\partial z} \, d\overline{z} \quad \text{or} \quad i \frac{1}{2} \overline{z} a(r) = \frac{\partial \phi}{\partial \overline{z}}.
$$

Now fix one circle, say the unit circle $C_1 \subseteq C^*$ and let $C_R = \{ z \in C^* | |z| = R \}$. Assume $R > 1$; $C_1$ and $C_R$ bound an annulus $\tilde{A}$ with $\partial \tilde{A} = C_R - C_1$. Now
\[
\int_A \int_A A^* \mu = \int_A \int_A \frac{i}{z} a(r) \, dz \wedge d\bar{z} = -\int_A \int_A \frac{\partial \phi}{\partial z} \frac{d\bar{z}}{z} \wedge dz
\]

\[
= -\int_{C_R} \phi \frac{dz}{z} + \int_{C_1} \phi \frac{dz}{z} = \frac{1}{i} \int_0^{2\pi} \phi(Re^{i\theta}) \, d\theta - C
\]

with \( C \) a constant. Now \( \int_0^{2\pi} \phi(Re^{i\theta}) \, d\theta \leq M' < \infty \) since \( \phi \) is the pullback of a bounded function on \( X \).

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Therefore \( \int_A A^* \mu \leq M'' < \infty \) where \( M'' \) is a positive constant independent of \( R \). Thus by Bishop's extension theorem (cf. [1], [2]), \( A \) extends holomorphically over \( \infty \). An identical argument gives extension at \( 0 \). Q.E.D.

Using the above Lemma and the Levi-Griffiths-Shiffman-Siu extension theorem (cf. [2], [9], [10], [11]) repeatedly, one proves the result for \( \text{SL}(2, \mathbb{C}) \) and groups of the form \( (\mathbb{C}^*)^n \) that have a fixed point on \( X \). Then one proves it for one parameter unipotent subgroup of \( G \) by using the above \( \text{SL}(2, \mathbb{C}) \) result on an \( \text{SL}(2, \mathbb{C}) \) in \( G \) containing the subgroup; this can be done by Jacobson-Morosow (cf. [4]). One now proves it for a Borel subgroup of \( G \) and uses an argument depending on the fact that one has a locally trivial fibering of \( G \) over \( G/B \) which is compact.

In the very interesting paper [6] of Lieberman, related matters are discussed.

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**BIBLIOGRAPHY**


