ON CUSPIDAL REPRESENTATIONS OF
p-ADIC REDUCTIVE GROUPS

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Abstract. Let $k$ be a $p$-adic field, and $G$ a reductive connected algebraic group over $k$. Fix a maximal torus $T$ of $G$ which splits in an unramified extension of $k$, and which has the same split rank as the center of $G$. For each character $\theta$ of $T(k)$, satisfying some conditions, there is a cuspidal representation $\gamma_\theta$ of $G(k)$ which is a sum of a finite number of irreducible representations; the correspondence $\theta \mapsto \gamma_\theta$ is one-to-one on the orbits of such characters by the little Weyl group of $T$; furthermore, the formulas for the formal degree of $\gamma_\theta$ and its character for sufficiently regular elements of $T(k)$ are given: they are formally the same as is the discrete series for real reductive groups.

1. Unramified maximal tori. Let $k$ be a $p$-adic field, that is a finite extension of $\mathbb{Q}_p$ or a field of formal series over a finite extension of $\mathbb{F}_p$. We denote by $\overline{k}$ the residue field of order $q$.

Let $G$ be a reductive connected algebraic group defined over $k$, the derived group $G_{\text{der}}$ of which is simply connected. A maximal torus of $G$ defined over $k$ is called minisotropic if it normalizes no (proper) horocyclic subgroup of $G$ defined over $k$.

**Lemma.** Suppose there exists a minisotropic maximal torus $T$ of $G$ which splits in a finite unramified extension $L$ of $G$. Then the Galois group $\Gamma$ of $L$ over $k$ has a unique fixed point $v$ in the apartment of $T$ in the building of $G_{\text{der}}(L)$ [2]; moreover, the face of $v$ is minimal amongst the faces in this apartment which are invariant by $\Gamma$.

2. Characters. We conserve notations and hypotheses of §1 and the Lemma. Let $\theta$ be a continuous character of $T(k)$. For each $\lambda \in \mathcal{X}^c(T)$, the lattice of rational one-parameter subgroups of $T$, we define a character $\theta_\lambda$ of $L^x$ by


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\[ \theta_{\lambda}(z) = \theta\left( \prod_{\Gamma}(z^\lambda) \right), \quad z \in L^x. \]

**Definition 1.** The character \( \theta \) is called regular if, for every root \( \alpha \) of \((G, T)\), the character \( \theta_{\alpha^\vee} \) of \( L^x \) associated to the coroot \( \alpha^\vee \) is nontrivial.

For each root \( \alpha \), we denote by \( |\alpha|_\theta \) the conductor of \( \theta_{\alpha^\vee} \) and let \( R_f \) be the set of roots of \((G, T)\) such that \( |\alpha|_\theta \leq f \).

**Definition 2.** The character \( \theta \) is called good if it is regular and if, for every \( f \), the set \( R_f \) is a convex set of roots.

We need the following form of Macdonald's conjecture (cf. [1, C-6.7]):

Let \( \overline{S} \) be a reductive connected algebraic group over \( \overline{k} \), \( \overline{T} \) a minisotropic maximal torus of \( \overline{S} \); fix a finite extension \( \overline{L} \) of \( \overline{k} \) which splits \( \overline{T} \); let \( \Gamma \) be its Galois group. A character \( \overline{\theta} \) of \( \overline{T}(\overline{k}) \) is called regular if, for every root \( \alpha \) of \((\overline{S}, \overline{T})\), the character \( z \mapsto \overline{\theta}(\prod_{\Gamma}(z^\alpha)) \) of \( L^x \) is nontrivial; if \( \overline{\theta} \) is a regular character of \( \overline{T}(\overline{k}) \), then there exists a unique class \( \overline{\sigma}_\theta \) of representations of \( \overline{S}(\overline{k}) \) such that, if \( \text{St}_S \) denotes the Steinberg representation of \( \overline{S}(\overline{k}) \), \( \text{St}_S \otimes \overline{\sigma}_\theta \) is the induced representation \( \text{Ind}(\overline{S}(\overline{k}), \overline{T}(\overline{k}), \overline{\theta}) \). Moreover \( \overline{\sigma}_\theta \) is cuspidal, and the intertwining number of two such representations \( \overline{\sigma}_{\theta_1} \) and \( \overline{\sigma}_{\theta_2} \) is equal to the number of elements in the little Weyl group of \( \overline{T} \) in \( \overline{G}(\overline{k}) \) which send \( \overline{\theta}_1 \) on \( \overline{\theta}_2 \).

3. Cuspidal representations.

**Theorem.** Let \( G \) be a reductive connected algebraic group over the \( p \)-adic field \( k \), the derived group \( G_{\text{der}} \) of which is simply connected. Let \( T \) be a minisotropic maximal torus which splits in an unramified extension of \( k \). Fix a good character \( \theta \) of \( T(k) \). Assume Macdonald's conjecture and one of the following conditions:

(i) \( |\alpha|_\theta > 1 \) for every root \( \alpha \) of \((G(L), T(L))\)

(ii) the residual characteristic of \( k \) is not 2 and there exists a rational representation \( \rho \) of \( G_{\text{der}} \) such that the corresponding bilinear form \( \text{Tr} \rho(X)\rho(Y) \) on the Lie algebra \( \text{Lie} \overline{T}_{\text{der}}(\overline{k}) \) is nondegenerate.

Then there exists a class \( \gamma_{\theta} \) of representations of \( G(k) \) such that:

(a) \( \gamma_{\theta} \) is a finite sum of irreducible representations of \( G(k) \), the coefficients of which have compact support modulo the center;

(b) the intertwining number of two such representations \( \gamma_{\theta_1} \) and \( \gamma_{\theta_2} \) is the number of elements in the little Weyl group \( W(T) \) of \( T \) in \( G(k) \) which send \( \theta_1 \) on \( \theta_2 \);
(c) there exists a Haar measure on $G(k)$, independent of $\theta$, such that the formal degree of $\gamma_\theta$ is

$$d(\theta) = \left( \prod_R q^{|\alpha|_\theta - 1} \right)^{1/2}$$

where $R$ is the set of roots of $(G, T)$;

(d) for $t \in T(k)$ such that $\text{val}(t^\alpha - 1) \geq |\alpha|_\theta / 3$ for every $\alpha \in R$, the value on $t$ of the character of $\gamma_\theta$ is given by

$$\text{Tr} \, \gamma_\theta(t) = (-1)^{l(G)}(-1)^{\Sigma_R / \Gamma (|\alpha|_\theta - 1)} \sum_{\Delta W(T)} \theta(wt)$$

where $l(G)$ is the split semisimple rank of $G$, $\Delta$ is the $W(T)$-invariant function on the regular elements of $T(k)$ given by

$$\Delta(t) = (-1)^{\Sigma_R / \Gamma \text{val}(t^\alpha - 1)} |\text{Det}_{\text{Lie} G / \text{Lie} T}(\text{Ad} \, t - 1)|^{1/2},$$

and $\Gamma$ is the Galois group over $\overline{k}$ of an unramified extension which splits $T$.

4. Remarks. 1. The proof is based upon an explicit construction of $\gamma_\theta$ (assuming Macdonald’s conjecture), obtained by inducing a finite dimensional representation of a compact open subgroup of $G(k)$ naturally associated to $T$ and $\theta$; the essential tool is given by Weil’s paper about Heisenberg groups [5]; we used too an argument given by R. Howe [4].

2. In the case where $|\alpha|_\theta$ is constant and strictly greater than 1, and if the point $v$ of the lemma is special, the proofs are given in [3].

3. G. Lusztig has just proved Macdonald’s conjecture.

REFERENCES


4. R. Howe, Tamely ramified supercuspidal representations of $GL_n$ (preprint).


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