THE INVALIDITY OF THE CALDERON-ZYGMUND INEQUALITY FOR SINGULAR INTEGRALS OVER LOCAL FIELDS

BY JAMES E. DALY

Communicated March 20, 1975

We will show that the Calderón-Zygmund inequality, \( \| T_\omega \|_p \leq C(p, r) \| \omega \|_r \), is not valid in the local field setting. A complete proof of the validity of this inequality in the case of singular integrals over \( \mathbb{R}^n \) can be found in Dunford and Schwartz, Linear operators, Vol. 2. We use the theory of regular functions as developed by M. Taibleson [4] and the F. and M. Riesz theorem for local fields as proved by J. Chao [1].

We assume the reader is familiar with elementary local field analysis and singular integrals in general. In the following work \( K \) will denote a local field (non-discrete, zero-dimensional, locally compact field), \( B^n = \{ x \in K: \| x \| \leq q^{-n} \}, D^n = \{ x \in K: \| x \| = q^{-n} \} \), and \( \xi_A \) the characteristic function of the set \( A \). Haar measure \( \lambda \) is normalized so that \( \lambda(B^0) = 1 \) (\( \lambda(B^1) = q^{-1} \)) and the prime \( \pi \) is chosen so that \( \pi B^0 = B^1 \). The fundamental character \( \chi \) is trivial on \( B^0 \) and nontrivial on \( B^1 \). \( C_0 \) and \( C_0 \) denote the continuous functions with compact support and the continuous functions that vanish at infinity, respectively.

**Definition.** For \( x \in k, k \in \mathbb{Z} \), let
\[
f(x, -k) = \begin{cases} 0, & k < 2, \\ \xi_{D0}(x) \sum_{j=2}^{k} \chi(\pi^{-j}x) & \text{if } k \geq 2. \end{cases}
\]

**Lemma 1.** The function \( f \) defined above is regular.

**Proof.** A function \( g: K \times Z \rightarrow C \) is said to be regular if
\[
g(x, k) = q^{-k} \int_{B^{-k}} g(y - x, k - 1) \, dy.
\]
A straightforward calculation shows that \( f \) satisfies this equality. \( \square \)

**Lemma 2.**

(a) \( \| f(-k) \|_2 = \frac{(q - 1)(k - 1)}{q} \) for \( k \geq 2 \).

(b) \( \| f(-k) \|_r \leq \frac{(q - 1)(k - 1)}{q}^{(r-1)/r} \) for \( k \geq 2 \), \( 2 < r < \infty \).


Key words and phrases. Singular integral, local field, Calderón-Zygmund inequality.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Copyright © 1975, American Mathematical Society
PROOF. 
\[ \hat{f}(x, -k) = \int_{D^0} f(y, -k) \overline{\chi(xy)} \, dy \]
\[ = \int_{B^0} \sum_{j=2}^{k} \chi((\pi^{-j} - x)y) \, dy - \int_{B^1} \sum_{j=2}^{k} \chi((\pi^{-j} - x)y) \, dy \]
(a)
\[ = \sum_{j=2}^{k} \xi_{\pi^{-j} + B^0}(x) - \frac{1}{q} \sum_{j=2}^{k} \xi_{\pi^{-j} + B^{-1}}(x) \]
\[ = \frac{q-1}{q} \sum_{j=2}^{k} \xi_{\pi^{-j} + B^0}(x) - \frac{1}{q} \sum_{j=2}^{k} \xi_{\pi^{-j} + D^{-1}}(x). \]

(b) \[ \|f(\cdot, -k)\|_2 = \|\hat{f}(\cdot, -k)\|_2 = \left\{ (q-1)(k-1)/q \right\}^{1/2} \]
by (a).

(c) Let \( 2 < r < \infty \) and \( f(\cdot, -k), \hat{f}(\cdot, -k) \in C_{00} \), \( \|f(\cdot, -k)\|_r \leq \|\hat{f}(\cdot, -k)\|_{r'} \)
where \( r' = r/(r-1) \). By (a), \( \|f(\cdot, -k)\|_r \leq \left\{ (q-1)(k-1)/q \right\}^{(r-1)/r} \).

DEFINITION. For \( k \geq 2 \), let \( \Gamma_k \) denote the set of \( \omega: K^* \to \mathbb{C} \) such that
\( \omega(x + B^k) = \omega(x) \) for \( x \in D^0 \)
\( \omega(\pi^j s) = \omega(x) \) for \( x \in K^*, j \in \mathbb{Z} \),
\( \int_{D^0} \omega(x) \, dx = 0 \),
and \( \Gamma = \bigcup_{k=2}^{\infty} \Gamma_k \).

We note each \( \omega \in \Gamma \) is the kernel of a singular integral operator \( T_\omega \) (see [3]). These kernels correspond to \( C^\infty \) kernels in the real case. We denote the multiplier of the operator \( T_\omega \) by \( F(T_\omega) \) and \( L_p \)-operator norm of \( T_\omega \) by \( \|T_\omega\|_p \).

By \( \|\omega\|_r \), we mean \( \{ \int_{D^0} |\omega(x)|^r \, dx \}^{1/r} \) if \( 1 \leq r < \infty \) and \( \sup_{x \in D^0} |\omega(x)| \) if \( r = \infty \).

LEMMA 3. If \( \omega \in \Gamma_k \), then
\[ \hat{f}(x)(y) = \int_{D^0} \omega(x) \sum_{j=1}^{k} \overline{\chi(\pi^{-j} xy)} \, dx \quad \text{for} \quad y \in D^0. \]
Consequently \( F(T_\omega) \) is constant upon the cosets of \( B^k \) in \( D^0 \).

PROOF. See [2, Proposition 1 and Corollary 2].

For the next theorem we preclude the case of even \( q \). The F. and M. Riesz theorem as proved by J. Chao requires this restriction.

THEOREM 1. For \( 1 < p < \infty, 1 < r < \infty \), there is no constant \( C(p, r) \) such that for all \( \omega \in \Gamma \),
\[ \|T_\omega\|_p \leq C(p, r) \|\omega\|_p. \]

PROOF. From the inequalities \( \|T_\omega\|_2 \leq \|T_\omega\|_p \) and \( \|\omega\|_p \leq \|\omega\|_\infty \), we need
only show there is no constant \( C \) such that \( \|T_\omega\|_2 \leq \|C\|\omega\|_\infty \) for all \( \omega \in \Gamma \). We
will accomplish this if we find a sequence \( \{\omega_k\} \subset \Gamma \) such that \( \|\omega_k\|_\infty \leq 2 \) and
\[ \|T_{\omega_k}\|_p = \|F(T_{\omega_k})\|_\infty \geq \|F(T_{\omega_k})(1)\|_\infty \to \infty. \]
To this end we define for \( x \in D^0 \),
\[ \omega_k(x) = \frac{1}{k} \sum_{j=1}^{k} \chi(\pi^{-j} xy) \]
for \( y \in D^0 \) and \( k \) large. Then
\[ \|\omega_k\|_\infty \leq 2 \]
and
\[ \|T_{\omega_k}\|_p \geq \frac{1}{k^{1/p}} \int_{D^0} \left| \sum_{j=1}^{k} \chi(\pi^{-j} xy) \right| \, dx \to \infty. \]
\[ g(x, -k) = \begin{cases} \frac{f(x, -k)}{|f(x, -k)|} & \text{if } f(x, -k) \neq 0, \\ 0 & \text{if } f(x, -k) = 0, \end{cases} \]

and

\[ \omega_k = g(x, -k) - \frac{q}{q - 1} \int_{D^0} g(x, -k) \, dx. \]

By the above definition, \( \int_{D^0} \omega_k(x) \, dx = 0 \) and \( \|\omega_k\|_\infty \leq 2 \). Thus if we extend \( \omega_k \) to \( K^* \) by homogeneity, \( \omega_k \in \Gamma_k \). We have

\[ F(T_{\omega_k})(1) = \int_{D^0} \omega_k(x) \sum_{j=1}^{k} \chi(\pi^{-j}x) \, dx \]

\[ = \int_{D^0} \left| \sum_{j=2}^{k} \chi(\pi^{-j}x) \right| \, dx + O(1) = \|f(\cdot, -k)\|_1 + O(1). \]

By Lemma 1 the function \( f \) is regular. Thus if \( \|f(\cdot, -k)\|_1 \leq A < \infty \), then \( f \) is the regularization of a finite Borel measure \( \mu \) [4, Theorem 8]. From Lemma 2(a) and the fact that \( f(\cdot, -k) \rightharpoonup \mu \) in the weak\(^*\)-topology of the dual of \( C_{00} \),

\[ \hat{\mu} = \frac{q - 1}{q} \sum_{j=2}^{\infty} \xi_{\pi^{-j}B^0} - \frac{1}{q} \sum_{j=2}^{\infty} \xi_{\pi^{-j}+D^{-1}}. \]

Thus \( \hat{\mu} \) is supported on the cone \( \sum_{j=-\infty}^{\infty} \pi((1 + B^1) \mathbb{Z}) \) and, therefore, \( \mu \) is absolutely continuous with respect to Haar measure [1, Corollary 5.3]. So \( \mu \) is given by an \( L_1 \)-function \( h \) and \( \hat{h} = \hat{\mu} \). But \( \hat{\mu} \) is not in \( C_0 \), so \( h \) cannot be in \( L_1 \). This contradiction gives \( \|F(T_{\omega})(1)\| \to \infty \). \( \square \)

We now give a variation of Theorem 1 by excluding the case \( r = \infty \) in (1) and allowing \( K \) to be any local field (no restriction on \( q \)). The proof is greatly simplified and depends only on Lemma 2.

**Theorem 2.** For \( 1 < p < \infty \), \( 1 < r < \infty \), there is no constant \( C(p, r) \) such that for all \( \omega \in \Gamma \), \( \|T_{\omega}\|_p \leq C(p, r)\|\omega\|_p \).

**Proof.** As in the proof of Theorem 1, we may restrict ourselves to \( \|T_{\omega}\|_2 \).

Let \( \beta_k(x) = f(x, -k) \) for \( x \in D^0 \) and extend \( \beta_k \) to \( K^* \) by homogeneity. Observe \( \beta_k \in \Gamma_k \). We have

\[ F(T_{\beta_k})(1) = \int_{D^0} \beta_k(x) \sum_{j=1}^{k} \chi(\pi^{-j}x) \, dx \]

\[ = \int_{D^0} \left| \sum_{j=2}^{k} \chi(\pi^{-j}x) \right|^2 \, dx + \int_{D^0} \sum_{j=2}^{k} \chi((\pi^{-j} - \pi^{-1})x) \, dx \]

\[ = \|f(\cdot, -k)\|_2^2 = (q - 1)(k - 1)/q \]

by Lemma 2(b). Thus \( \|T_{\beta_k}\|_2/\|\beta_k\|_r \to \infty \) by Lemma 2(c). For \( 2 \leq r < \infty \), this rate of growth is greater than \( (q - 1)(k - 1)/q \)^{1/r}. \( \square \)

Theorem 2 is valid in the \( n \)-dimensional case with no changes except for some constants in the proof. The proof of Theorem 1 can be used except one must find another method to show \( \|f(\cdot, -k)\| \to \infty \), as the work of Chao is for
One would expect \( \|f(\cdot, -k)\|_1 = o(\log k) \) as in the case of the classical Dirichlet kernel. A direct computation in the case \( k \) is the 2-series field substantiates this conjecture.

REFERENCES


DEPARTMENT OF MATHEMATICAL SCIENCES, NEW MEXICO STATE UNIVERSITY, LAS CRUCES, NEW MEXICO 88003