IDEALS AND POWERS OF CARDINALS

BY KAREL PRIKRY

Communicated by Alistair Lachlan, April 28, 1975

We obtain results concerning the behaviour of the function \(2^{\omega \alpha} \ (\alpha \in \text{On})\) under the assumption of the existence of certain kind of ideals. These results complement those of Ulam [7], Tarski [6] and Solovay [4] and [5]. In particular, it follows that if \(2^\omega\) is real-valued measurable, then \(2^\nu = 2^\omega\) for all infinite \(\nu < 2^\omega\).

We assume some familiarity with [4] and [5]. \(\alpha, \beta, \gamma, \delta, \eta, \xi, \rho \ (\kappa, \lambda, \nu, \tau)\) denote ordinals (inf. cardinals). \(f, g, h\) denote functions; \(F\) denotes families of functions or sets. We use the Erdös-Hajnal notation \([S]^{\nu}, [S]^{<\nu}\), etc. (see [2]). \(F\) is \(\lambda\)-almost disjoint (\(\lambda\)-a.d.) if \(|X \cap Y| < \lambda\) whenever \(X, Y \in F\) and \(X \neq Y\).

DEFINITION 1. \(\kappa\) is \(\lambda\)-real-supercompact (abbrev. \(\lambda\)-r.s.c.) if there is a real-valued \(\kappa\)-compl. measure \(\mu\) defined on \(P([\lambda]^{<\kappa})\) such that

(i) \(\mu([\lambda]^{<\kappa}) = 1\);

(ii) for every \(\alpha \in \lambda\), \(\mu(\{x: \alpha \notin x\}) = 0\);

(iii) if \(\mu(X) > 0\) and \(f: X \to \lambda\) is such that \(f(\xi) \in x\) for all \(x \in X\), then there is \(Y \subseteq X\) such that \(\mu(Y) > 0\) and \(f\) is constant on \(Y\).

\(\kappa\) is r.s.c. if \(\kappa\) is \(\lambda\)-r.s.c. for all regular \(\lambda \geq \kappa\). We define "\(\kappa\) is \(\omega_1\)-saturatedly supercompact" (abbrev. \(\omega_1\)-s.s.c.) by replacing \(\mu\) by an ideal \(I\) in the obvious way.

One can show by the methods of [3] and [4] that if it is consistent that a s.c. cardinal exists, then it is consistent that \(2^\omega\) is r.s.c.

DEFINITION 2. \(R_2(\kappa_0, \kappa_1)\) holds if for every partition \([\kappa_1]^2 = \bigcup\{K_\xi: \xi \in \lambda\}\), where \(\omega < \lambda < \kappa_0\), there exists an \(X \subseteq \kappa_1\) and \(M \subseteq \lambda\) such that \(|X| = \kappa_0\), \(|M| < \lambda\), and \([X]^2 \subseteq \bigcup\{K_\xi: \xi \in M\}\).

THEOREM 1. Let \(\lambda, \nu < \kappa\), \(\omega < \text{cf}(\lambda)\) and \(F \subseteq [\nu]^{>\lambda}\) be \(\lambda\)-a.d. If \(R_2(\kappa, \kappa)\) holds and \(\text{cf}(\kappa) > \omega\), then \(|F| < \kappa\). If \(R_2(\kappa, \kappa_1)\) holds and \(\kappa_1\) is regular, then \(|F| < \kappa_1\).

THEOREM 2. Set \(2^\omega = \kappa\) and suppose that \(\kappa\) carries a \(\kappa\)-compl. \(\omega_1\)-sat. nontrivial ideal. Then

(a) for all \(\nu < \kappa\), \(2^\nu = \kappa\);

(b) if \(I \subseteq \mathcal{P}(\kappa)\) is \(\omega_1\)-compl., \(\omega_1\)-sat. and \([\kappa]^{<\kappa} \subseteq I\), then \(|\mathcal{P}(\kappa)/I| = 2^\kappa\);

(c) if \(\nu < \kappa\) and \(\text{cf}(\nu) > \omega\), then there is a family \(F \subseteq [\nu]^\nu\) such that \(|F| < \kappa\) and each \(g \in [\nu]^\nu\) is dominated everywhere by some \(f \in F\).
(d) if $\lambda, \nu < \kappa$, $\omega < \text{cf}(\lambda)$ and $F \subseteq [\nu]^{>\lambda}$ is $\lambda$-a.d., then $|F| < \kappa$.

**Theorem 3.** Suppose that $2^\omega = \kappa$ is $\omega_1$-s.s.c. Then

(a) $\lambda^\kappa = \lambda$ for all regular $\lambda > \kappa$;
(b) $2^\nu = \nu^+$ for all singular strong limit $\nu > \kappa$;
(c) if $I \subseteq \mathcal{P}(\kappa)$ is $\omega_1$-compl., $\omega_1$-sat., $[\kappa]^{<\kappa} \subseteq I$ and $\mathcal{P}(\kappa)/I$ can be generated (by infinitary Boolean operations) from $\lambda$ elements, then either $2^\kappa = \lambda$, or $2^\kappa = \lambda^+$ and $\text{cf}(\lambda) = \omega$;
(d) if $\lambda \geq \kappa$, then $\square^\lambda_\lambda$ is false (see [5] for the statement of $\square^\lambda_\lambda$).

Solovay [4, Lemma 14, p. 406] proved that $R_2(\kappa, \kappa)$ holds if $\kappa$ carries a $\kappa$-compl. $\omega_1$-sat. nontrivial ideal. The proof of Theorem 2(a) uses this result, Theorem 1, and Tarski’s “almost disjoint sets” construction. It proceeds by induction on $\nu < 2^\omega$.

Theorem 2(b) strengthens a result of Kunen who showed that $|\mathcal{P}(\kappa)/I| \geq \kappa^+$. To prove this, he used the fact that in the Boolean-valued universe $V^{\mathcal{P}(\kappa)/I}$, $|\mathcal{P}(\omega)| \geq \kappa^+$. Theorem 2(a) enables us to show that in $V^{\mathcal{P}(\kappa)/I}$, $|\mathcal{P}(\omega)| = 2^\kappa$.

To prove Theorem 2(c), we again use a method of Kunen who showed that the corresponding result holds for $\omega \omega$ if $2^\omega$ is r.v.m. This is made possible by Theorem 2(a). The method involves considering Solovay’s Boolean ultrapower $V^\kappa/I$.

The proof of Theorem 3 involves ideas of [5, §§3 and 4] and an additional unpublished result of Solovay.

**Lemma 1 (Solovay, unpublished).** For every regular $\lambda > \omega$ there exists an $\omega$-ary Jónsson algebra $(\lambda, f)$ such that for every $X \subseteq \lambda$, $|\text{rng}(f \restriction [X]^{\omega})| \leq |X|$.

**Lemma 2.** Let $\lambda \geq \kappa$ be regular and $\mu$ be a measure as in Definition 1.

(a) If $X \subseteq [\lambda]^{<\kappa}$ and $\mu(X) = 1$, then $|X| = \lambda^\kappa$.
(b) Let $g$: $[\lambda]^{<\kappa} \rightarrow \lambda$ be defined by $g(x) = \sup(x)$. Then there is $X \subseteq [\lambda]^{<\kappa}$ such that $\mu(X) = 1$ and $g \upharpoonright X$ is one-to-one.

The proof of Lemma 2(a) uses Theorem 2(a). Lemma 2(b) is analogous to Theorem 2 of [5]. The proof of Lemma 2(b) uses Lemma 1 where Solovay’s proof of his Theorem 2 used an older result of [1]. Some modifications are required and this holds for the proof of Theorem 3(d) as well. Theorem 3(a) follows from Lemma 2 and implies Theorem 3(b). Theorem 3(c) follows from Theorem 2(b) and Theorem 3(a).

The author wishes to thank K. Kunen for valuable discussions concerning the subject of this note.

**References**


License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455