CONCAVITY OF MAGNETIZATION FOR A CLASS OF EVEN FERROMAGNETS

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1. Introduction. Let $E$ be the set of even probability measures which satisfy $f \exp(kx^2)p(dx) < \infty$ for all $k > 0$ sufficiently small. Given an integer $N \geq 1$, real numbers $h > 0$ and $J_{ij} \geq 0$, $1 \leq i \leq j \leq N$, and measures $\rho_i \in E$, $1 \leq i \leq N$, we define [11, p. 273] real-valued random variables $X_i$, $1 \leq i \leq N$, with the joint distribution

$$
\tau_h(dx_1, \ldots, dx_N) = \frac{\exp(\sum_{1 \leq i < j \leq N} J_{ij}x_i x_j + h\sum_{1 \leq i \leq N} x_i) \rho_1(dx_1) \cdots \rho_N(dx_N)}{Z(h)}.
$$

$Z(h)$, the partition function, is given by the formula

$$
Z(h) = \int \cdots \int_R \exp\left(\sum_{1 \leq i < j \leq N} J_{ij}x_i x_j + h\sum_{1 \leq i \leq N} x_i\right) \rho_1(dx_1) \cdots \rho_N(dx_N).
$$

The $J_{ij}$ are assumed to be so small that the integral in (2) converges for all $h > 0$. The inequalities we discuss are to hold for all $h > 0$ and all $J_{ij} \geq 0$ subject only to this restriction. The choice of $\rho_i$ as the Bernoulli measure $b(dx) = \frac{1}{2}(\delta(x - 1) + \delta(x + 1))$ gives the classical Ising model.

We define the average magnetization per site, $m(h)$, by the formula

$$
m(h) = \frac{1}{N} \frac{d}{dh} \ln Z(h) = \frac{1}{N} \sum_{i=1}^{N} E[X_i],
$$

and consider inequalities on $m(h)$ and its derivatives. While the inequalities $m(h) \geq 0$, $dm(h)/dh \geq 0$ hold for any $\rho_i \in E$ [7, pp. 76–77], the concavity of $m(h)$, i.e.

$$
d^2m(h)/dh^2 \leq 0,
$$

requires that further restrictions be placed on the $\rho_i$. Essentially, (4) is known to hold only in the Ising case and in models which can be built out of Ising models in a suitable way [4], [6]. Measures for which (4) fails are known [6].

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The usual approach to (4) is first to prove the stronger (GHS) inequalities

\[ \frac{\partial^3}{\partial h_i \partial h_j \partial h_k} \ln Z(h_1, \ldots, h_N) \leq 0, \quad \text{all } 1 \leq i, j, k \leq N, h_i \geq 0, \]

where

\[ Z(h_1, \ldots, h_N) = \int_{\mathbb{R}^N} \int \exp \left( \sum_{i,j} J_{ij} x_i x_j + \sum h_i x_i \right) \rho_1(dx_1) \cdots \rho_N(dx_N). \]

Instead, we shall prove (4) directly for many new measures using a technique which reduces consideration to the case \( N = 1 \). Afterwards, we shall return to (5).

We state two implications of these inequalities. The first shows that the requirement that the \( \rho_i \) in (1) have Gaussian falloff is only an apparent restriction.

**Theorem 1.** Let \( \rho \) be an even probability measure satisfying \( \int \exp(kx) \rho(dx) < \infty \) for all \( k \geq 0 \). Assume that (4) holds for \( N = 1 \) (set \( \rho_1 = \rho \)). Then \( \rho \) is in \( \mathcal{E} \).

The next theorem (known for fourth degree polynomial \( V \) [3], [10]) on the spectrum of certain differential operators is a striking consequence of (5).

**Theorem 2.** Let \( V(x) \) be an entire function with the expansion

\[ V(x) = \sum_{k=1}^{\infty} a_k x^{2k}, \quad a_k \geq 0 \quad \text{for } k \geq 2, \quad a_1 \text{ real} \quad (a_1 > 0 \text{ if all } a_k = 0). \]

Let \( E_1, E_2, E_3 \), be the three smallest eigenvalues of the differential operator 
\(- \frac{\partial^2}{\partial x^2} + V(x)\) on \( L^2(R^1; dx) \). Then \( E_3 - E_2 \geq E_2 - E_1 \).

By Theorems 4 and 5 below, we shall see that (5) is satisfied for the measures

\[ \rho_i(dx) = c \exp(-V(x)) dx, \quad c \text{ a normalization constant}, \]

if \( V \) is as in (6). This is the main ingredient needed to prove Theorem 2 [10].

2. The class \( \mathcal{G}_- \). Below, we define a subset \( \mathcal{G}_- \) of measures in \( \mathcal{E} \) for which we have the following result.

**Theorem 3.** If \( \rho_1, \ldots, \rho_N \in \mathcal{G}_- \), then (4) holds.

For the proof, we use a closure property of \( \mathcal{G}_- \) in order to reduce to the case \( N = 1 \). We call this property the closure of \( \mathcal{G}_- \) under ferromagnetic unions.

(C) Let \( Y_1, \ldots, Y_N \) be real-valued random variables with joint distribution \( \tau_0 \) (see (1)). Let \( \mathcal{F}_0 \) be the class of all distributions of sums \( \sum_{1 \leq i \leq N r_i} Y_i \).
for arbitrary choice of \( N \geq 1, r_i \geq 0, J_{i,j} \geq 0, \) and \( \rho_1, \ldots, \rho_N \in G_\text{.} \) Then \( F_0 \subseteq G \).

The partition function \( Z(H) \) in (2) can be written as

\[
Z(h) = Z(0)E\left\{ \exp \left( h \sum_{1 \leq i \leq N} Y_i \right) \right\};
\]

i.e., \( m(h) \) is related to the average magnetization \( \bar{m}(h) \) for a single site system (with spin \( \sum_{1 \leq i \leq N} Y_i \) at the single site) by the formula \( m(h) = N^{-1} \bar{m}(h) \). Hence, Theorem 3 for general \( N \) is a consequence of (C) once we have proved Theorem 3 for \( N = 1 \). We do the latter in §3.

The next theorem indicates which measures belong to \( G_\text{.} \).

**Theorem 4.** \( G_\text{.} \) contains the Bernoulli measure \( b(dx) \) and all measures of the form (7), where \( V(x) \) is as in (6). Also, \( G_\text{.} \) contains the distributions of all weak limits of \( Y^{(N)} \in F_0 \) which satisfy \( \sup N E[(Y^{(N)})^2] < \infty \).

The first part of Theorem 4 will be proved after \( G_\text{.} \) is defined.

**Definition.** Given \( \rho \in E \), let \( W_1, \ldots, W_4 \) be four independent copies of a random variable distributed by \( \rho \). The vector \( \bar{m} = (m_1, \ldots, m_4) \), where each \( m_i \) is a nonnegative integer, is said to be odd if each \( m_i \) is odd. Let \( W = (W_1, \ldots, W_4) \); take \( A \) to be the orthogonal matrix

\[
2^{-1} \begin{pmatrix}
1 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 \\
-1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1
\end{pmatrix};
\]

and define

\[
(A\bar{W})_i = \sum_{j=1}^4 A_{ij} W_j, \quad (A\bar{W})^m = (A\bar{W})_1^{m_1} \cdots (A\bar{W})_4^{m_4}, \quad \mu_\rho(\bar{m}) = E[(A\bar{W})^m].
\]

We define

\[
G_\text{.} = \{ \rho : \rho \in E \text{ and } \mu_\rho(\bar{m}) \leq 0 \text{ for all } \bar{m} \text{ odd} \}.
\]

The condition that \( \mu_\rho(\bar{m}) \leq 0 \) for all \( \bar{m} \) odd implies an infinite string of inequalities satisfied by the moments of a measure \( \rho \in G_\text{.} \). We refer the reader to [8] and [9], where other moment inequalities are derived for measures which satisfy the Lee-Yang theorem.

To show that \( b(dx) \in G_\text{.} \), consider

\[
\mu_b(\bar{m}) = \frac{1}{16} \sum_{x_i = \pm 1} \frac{1}{x_1 + x_2 + x_3 + x_4}^{m_1}(-x_1 + x_2 - x_3 + x_4)^{m_2} \cdot (-x_1 - x_2 + x_3 + x_4)^{m_3}(x_1 - x_2 - x_3 + x_4)^{m_4}.
\]

Each of the 16 summands is either negative or zero according to whether an odd number or an even number of the \( x_i \) equal +1.

Given a measure \( \rho \) as in (7), the joint distribution of the random vector
$A \tilde{W}$ has the form
\[ \exp(-f(z_1, \ldots, z_4)) \exp(g(z_1, \ldots, z_4)) dz_1 \cdots dz_4, \]
where $f$ is an odd function of each $z_i$ and $f \geq 0$ when each $z_i \geq 0$ and $g$ is an even function of each $z_i$. Greater weight is thus given to those values of $z_1, \ldots, z_4$ where an odd number of the $z_i$ are negative than where an even number of the $z_i$ are negative. From this, it can be shown that $\rho \in \mathcal{G}_-$. We also have a characterization of Gaussian measures in terms of $\mathcal{G}_-$.

**Theorem 5.** Given $\rho \in \mathcal{E}$, the numbers $\mu_{\rho}(\overline{m}) = 0$ for all $\overline{m}$ odd if and only if $\rho$ is an even Gaussian measure.

Inequality (5) holds under the same hypothesis as (4).

**Theorem 6.** If $\rho_1, \ldots, \rho_N \in \mathcal{G}_-$, then (5) holds.

The proof makes use of multivariate versions of the $\mathcal{G}_-$ inequalities. Let $Y_i^{(j)}$, $1 \leq j \leq 4$, be independent copies of $Y_i$ (see (C)) and define $\overline{Y}_1 = (Y_1^{(1)}, \ldots, Y_1^{(4)})$. Then
\[ E[(AY_1)^{\overline{m}_1} \cdots (A\overline{Y}_N)^{\overline{m}_N}] \leq 0 \]
whenever $\rho_1, \ldots, \rho_N \in \mathcal{G}_-$ and $\overline{m}_1 + \ldots + \overline{m}_N$ is odd.

3. **Proof of Theorem 3 for $N = 1$.** Given $\rho \in \mathcal{G}_-$, we write $Z(h) = \int \exp(hx) \rho(dx)$, $h \geq 0$. We have (‘ denotes $d/dh$)
\[ \frac{d^3}{dh_1^3} \frac{d^3}{dh_2^3} \frac{d^3}{dh_3^3} + 2 \frac{d^3}{dh_1^3} \frac{d^3}{dh_2^3} \frac{d^3}{dh_3^3} \int_{R^4} \rho(dx_1) \cdots \rho(dx_4) \bigg|_{h_i = h}, \]
where $\overline{h} = (h_1, \ldots, h_4)$, $\overline{x} = (x_1, \ldots, x_4)$, and $\langle \cdot, \cdot \rangle$ is the $R^4$ inner product. Define $\overline{s} = (s_1, \ldots, s_4) = \overline{h} A^t$. An easy calculation [1, Appendix] shows that the last integral equals
\[ \frac{2}{Z^4} \int_{R^4} \frac{d^3}{ds_2^3} \frac{d^3}{ds_3^3} \frac{d^3}{ds_4^3} e^{\langle \overline{z}, \overline{x} \rangle} \rho(dx_1) \cdots \rho(dx_4) \bigg|_{h_i = h}. \]

Expanding the exponential and carrying out the integration, we find
\[ \frac{d^3}{dh_1^3} \frac{d^3}{dh_2^3} \frac{d^3}{dh_3^3} \int_{R^4} \rho(dx_1) \cdots \rho(dx_4) \bigg|_{h_i = h}. \]

\[ \ln Z''' = \sum_{n=0}^{\infty} \sum_{m_1 + \ldots + m_4 = n} \frac{m_2 m_3}{m_1!} \frac{m_4}{m_4!} \mu_{\rho}(\overline{m}) s_1^{m_1} s_2^{m_2-1} s_3^{m_3-1} s_4^{m_4-1} \bigg|_{h_i = h}. \]
But when each \( h_i = h \), then \( s_1 = 2h, s_2 = s_3 = s_4 = 0 \). Also, \( \mu_\rho((k, 1, 1, 1)) \) can be shown to be zero unless \( k \) is odd. Hence

\[
(\ln Z)^{'''(k, 1, 1, 1)) = \frac{2}{Z^4} \sum_{k \text{ odd} : k \geq 0} \frac{(2h)^k}{k!} \mu_\rho((k, 1, 1, 1)),
\]

which is negative since \( \rho \in G_\_ \) and \( h \geq 0 \). This completes the proof.

In this proof, we did not need the full force of the assumption that \( \rho \in G_\_ \); viz., that \( \mu_\rho(\overline{m}) \leq 0 \) for all \( \overline{m} \) odd. However, the latter is needed to prove Theorem 6. Also, the set of measures \( \tilde{\rho} \) for which \( \mu_\tilde{\rho}((k, 1, 1, 1)) \leq 0 \) for all \( k \) odd is not necessarily closed under ferromagnetic unions.

**Remark.** Proofs and related matter will appear in [2].

**REFERENCES**

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