A PRESENTATION FOR SOME $K_2(n, R)$

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1. All rings are commutative with identity. We announce a presentation for the $K_2$ of a class of rings which includes the local ones. We also give a presentation for the relative $K_2$ of a homomorphism that splits and has its kernel in the Jacobson radical. These results generalize (and were suggested by) various earlier ones: the presentation of Matsumoto for the $K_2$ of (infinite) fields [6], [7, §11, 12]; the presentation of Dennis and Stein for the $K_2$ of discrete valuation rings and homomorphic images thereof [2]; stability results of the same authors [4]; the presentation for the relative $K_2$ of dual numbers, by one of us [5]. We reproved most of the earlier results and generalized them in the process.

2. The functor $D$ (cf. [3, §9]).

2.1. Let $R$ be a ring, $R^*$ its group of units. We define the abelian group

$$D(R)$$

by the following presentation:

Generators are the symbols $(a, b)$ with $a, b \in R$ such that $1 + ab \in R^*$.

Relations are: (D0) commutativity.

(D1) $(a, b) (-b, -a) = 1$.

(D2) $(a, b) (a, c) = (a, b + c + abc)$.

(D3) $(a, bc) = (ab, c) (ac, b)$.

In all of these relations it is assumed that the left-hand sides make sense. For instance, in (D3) one needs $a, b, c \in R$ with $1 + abc \in R^*$. $D$ is a functor from (commutative) rings to abelian groups. It commutes with finite direct products.

2.2. Put $K_2(n, R) = \ker(\text{St}(n, R) \to \text{SL}(n, R))$, so that $K_2(R) = \lim K_2(n, R)$. Put $K_2(\infty, R) = K_2(R)$. Relations (D1), (D2), (D3) imply the relations in [3, §9] and vice versa. So the rule

$$(a, b) \mapsto x_{12} \left( \frac{-b}{1 + ab} \right) x_{12}(a) x_{21}(b) x_{12} \left( \frac{-a}{1 + ab} \right) h_{12}^{-1}(1 + ab)$$

defines a homomorphism $D(R) \to K_2(R)$ factoring through $K_2(3, R)$.

2.3. DEFINITION. $R$ is called 3-fold stable if, for any triple of unimodular sequences $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ there exists $r \in R$ such that $a_i + b_i r \in R^*$ for $i = 1, 2, 3$. (Recall that $(a, b)$ is called unimodular if $aR + bR = R$.) Similar definitions can be given for $k$-fold stability, e.g., 1-fold stability is the strongest of Bass’ stable range conditions $SR_n(R)$ [1]. The condition of 3-fold stability is


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934
still stronger than that of 1-fold stability.

2.4. Theorem 1. Let \( R \) be local or 3-fold stable. Then \( D(R) \rightarrow K_2(n, R) \) is an isomorphism for \( 3 \leq n \leq \infty \).

2.5. Now let \( I \) be an ideal contained in the Jacobson radical \( \text{Rad}(R) \) of \( R \). The abelian group \( D(R, I) \) is defined by the following presentation:

Generators are the \( \langle a, b \rangle \) with \( a \in R, b \in I \) or \( a \in I, b \in R \).

Relations are: commutativity; (D1) for \( a \in I, b \in R \); (D2) for \( a \in R, b, c \in I \); (D2) and (D3) for \( a \in I, b, c \in R \). (See 2.1 and compare [8, §2].) As in 2.2, one has a homomorphism \( D(R, I) \rightarrow K_2(R) \). It factors through \( D(R) \).

2.6. Theorem 2. Let \( I \) be an ideal, contained in \( \text{Rad}(R) \), such that \( R \rightarrow R/I \) splits. Then

\[
1 \rightarrow D(R, I) \rightarrow K_2(n, R) \rightarrow K_2(n, R/I) \rightarrow 1
\]

is split exact for \( 3 \leq n \leq \infty \).

2.7. Theorem 3. Let \( f: R \rightarrow S \) be a homomorphism of rings inducing an isomorphism \( R/\text{Rad}(R) \rightarrow S/\text{Rad}(S) \). If \( 3 \leq n \leq \infty \) and \( D(R) \rightarrow K_2(n, R) \) is an isomorphism, then \( D(S) \rightarrow K_2(n, S) \) is an isomorphism.

2.8. Examples and Remarks. (1) A semilocal ring is \( k \)-fold stable if and only if all its residue fields contain at least \( k + 1 \) elements.

(2) The ring of continuous complex valued functions on a 1-dimensional complex is \( k \)-fold stable for any \( k \in \mathbb{N} \).

(3) The ring of all totally real algebraic integers in \( \mathbb{C} \) is \( k \)-fold stable for any \( k \in \mathbb{N} \) (H. W. Lenstra).

(4) If \( R \) is 5-fold stable, then we can also show that \( K_2(R) \) can be presented by Matsumoto's relations [3, §11]. For local rings with infinite residue fields the analogous result holds for any type of Chevalley group (cf. [6, Corollaire 5.11]).

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