ON A MEAN VALUE INEQUALITY

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In this note we discuss a mean value inequality satisfied by functions $u(x,t)$ defined in the half space $\mathbb{R}^{n+1}_+$ which are solutions of a partial differential equation of semielliptic type. We then apply this result to the study of spaces of non-isotropic Riesz potentials and to the determination of the classes which arise as traces of the functions $u(x,t)$. The justification for considering these functions lies in the fact that they are a natural substitute for harmonic functions when Laplace's equation is not satisfied and they are related to the study of singular integrals with mixed homogeneity. It is a pleasure to acknowledge the conversations we had with Dr. A. P. Calderón concerning these topics.

The mean value inequality. We let \( \{A_t\}_{t>0}, A_{ts} = A_t A_s \) be a continuous group of affine transformations of \( \mathbb{R}^n \) leaving the origin fixed and denote its infinitesimal generator by \( P \) so that \( t(d/dt)A_t = PA_t \). We further assume that \( (P_x, x) > (x, x) \) for \( x \in \mathbb{R}^n \) and associate to each group \( A_t \) a translation invariant distance function \( \rho(x) \) defined to be the unique value of \( t \) such that \( |A_t^{-1} x| = 1, \rho(0) = 0 \). To the transpose \( A_t^* \) of \( A_t \) we associate \( \rho^*(x) \) in a similar fashion. As is well known \( \det A_t = \det A_t^* = t^\gamma \), \( \gamma = \text{trace } P \) (see [5, §1.1]). For \( \alpha = (\alpha_1, \ldots, \alpha_k), 1 \leq \alpha_i \leq n, \) and \( x^1, \ldots, x^k \) in \( \mathbb{R}^n \) we let \( \xi = x^1 \otimes \cdots \otimes x^k \) to be the element with components \( \xi_\alpha = \Pi_{i=1}^k x_{\alpha_i}^i \). For \( n \times n \) matrices \( A_1, \ldots, A_k, \) we put \( (A_1 \otimes \cdots \otimes A_k)(x^1 \otimes \cdots \otimes x^k) = A_1 x^1 \otimes \cdots \otimes A_k x^k \) and abbreviate this by \( \otimes^k A x \) when \( A = A_1, x^i = x \) for \( 1 \leq i \leq k \).

\[ \partial = (\partial/\partial x_1, \ldots, \partial/\partial x_n), \partial/\partial t \text{ and } \otimes^k A \partial \text{ acting on functions } u(x, t) \text{ have the obvious meaning.} \]

We set \( p_j(t, \partial) = \otimes^k L A_t^* \partial, \) where \( L^2 = (P + P^*)/4\pi. \)

Given a function \( \psi(x) \) we define the dilations \( \psi_t(x) = t^{-\gamma} \psi(A_t^{-1} x) \). A special role is played by \( \varphi_t(x) \) with \( \varphi(x) = e^{-\pi|x|^2}. \) This particular function \( \varphi_t(x) \) satisfies a differential equation, as is readily seen by taking Fourier transforms, namely \( A \varphi_t(x) = 0 \) where

\[ A = \frac{\partial}{\partial t} - \frac{1}{2\pi t} (P^* A_t^* \partial, A_t^* \partial) = \frac{\partial}{\partial t} - \frac{1}{t} (L A_t^* \partial, L A_t^* \partial). \]

We also have \( A u(x, t) = 0, \) whenever \( u(x, t) = f * \varphi_t(x), f \in S'(R^n). \)

We now state the mean value inequality and give some applications in the following sections.

MEAN VALUE INEQUALITY. Let \( A u(x, t) = 0 \) and \( 0 \leq r \leq k, \) then

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\[ |p_k(t, \partial)u(x, t)|^q \leq c r^\gamma \int_{t/2}^t \int_{\rho(x-y) \leq t} |p_k(s, \partial)u(y, s)|^q \, dy \, ds , \]
for \(1 \leq q < \infty\).

**Nonisotropic Riesz potentials.** (See [1], [3], [7], [12], [18] and [20].) For a positive real number \(\alpha\) we define the Riesz potential \(I_\alpha\) of order \(\alpha\) of \(f\) by means of

\[ (I_\alpha f)(x) = \rho^* (x)^{-\alpha} f(x), \quad 0 < \alpha < \gamma, \]
and for \(1 < p < \infty\), the classes \(L^p_\alpha (\mathbb{R}^n) = \{ f \in L^p (\mathbb{R}^n) : f = I_\alpha \eta, \eta \in L^p (\mathbb{R}^n) \}\) and we set \(\| f \|_{p, \alpha} = \| f \|_p + \| \eta \|_p\).

We now consider the following variants of the Littlewood-Paley function to express the norm in \(L^p_\alpha\) by an equivalent quantity (see [4], [10], [14], [16], [17]). Let

\[ G_q(k, \alpha, \lambda, x) = \left( \int_0^\infty \int_{\mathbb{R}^n} \frac{|p_k(s, \partial)u(y, s)|^q}{s^{-\alpha q - \gamma}} \, dy \, ds \right)^{1/q} \]
for \(k \geq 1, 0 < \alpha < k\) and \(\lambda > 1\).

**Theorem.** Let \(u(x, t) = f(t)\); then \(f\) is in \(L^p_\alpha (\mathbb{R}^n)\) if and only if \(f\) is in \(L^p (\mathbb{R}^n)\) and \(G_2 (k, \alpha, \lambda, x)\) is in \(L^p (\mathbb{R}^n)\), provided \(\lambda > 2/p\), and \(\| f \|_{p, \alpha} \approx \| f \|_p + \| G_2 (k, \alpha, \lambda) \|_p\). Moreover, if \(q \geq 2\) and \(\lambda > q/p\), then \(\| G_q (k, \alpha, \lambda) \|_p \leq c \| f \|_{p, \alpha}\), and if \(\lambda = q/p\) and \(p < 2\) we have the weak-type inequality

\[ \{ x \in \mathbb{R}^n : G_q (k, \alpha, q/p, x) > \mu \} \leq c \| f \|^p_{p, \alpha}/\mu^p. \]

That such weak-type inequalities follow from results in [5, §3.3] was indicated to us by N. Aguilera.

Closely related to these questions are the functions \(D_\alpha^q\) and \(D_\alpha^p\) (see [2], [14], [15], [20]) defined as follows:

\[ D_\alpha^q (x) = \left[ \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(x-y) - f(x)|^q}{\rho(y)^{\gamma + \alpha q}} \, dy \right]^{1/q}, \]
\[ D_\alpha^p (x) = \left[ \int_0^\infty \frac{1}{t^q} \left\{ \int_{\rho(y) \leq 1} |f(x + Ay) - f(x)|^p \, dy \right\}^{q/p} \, dt \right]^{1/q} \]
where \(0 < \alpha < 1\) and \(1 \leq p, q < \infty\).

Indeed we have the following result.

**Theorem.** Let \(u(x, t) = f(t)\); then for \(p > 2\gamma/(\gamma + 2\alpha)\),

\[ \| f \|_{p, \alpha} \approx \| f \|_p + \| D_\alpha^2 \|_p \quad \text{and} \quad \{ x \in \mathbb{R}^n : D_\alpha^2 (x) > \mu \} \leq c \| f \|^p_{p, \alpha}/\mu^p \]
for \(p = 2\gamma/(\gamma + 2\alpha)\).

Also if \(1 \leq r < q < \infty\), \(q \geq 2\) and \(p > r\gamma/(\gamma + qr)\), then

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\[ \|D_{rq}^{\alpha}\|_p \leq c\|f\|_{p,\alpha}, \quad \text{and} \quad |\{x \in \mathbb{R}^n : D_{rq}^{\alpha}(x) > \mu\}| \leq c\|f\|_{p,\alpha}^p/\mu^p, \]

for \(1 < p = r\gamma/(\gamma + \omega) < 2\).

**Traces of the spaces \(H^{\alpha,p}\).** These results were obtained jointly with A. Ortiz and extend the interesting results of [6]. Let \(0 \leq \alpha < 1, 1 < p < \infty\). We say that \(u(x, t) \in H^{\alpha,p}\) if \(A(u(x, t) = 0\) and

\[ |u|_{p,\alpha} = \sup_{x,t} \left[ \frac{1}{t^{\gamma + \alpha p}} \int_{\rho(x-y) < t} \left[ \frac{1}{t} \int_0^t |p_1(s,\partial)u(y,s)|^2 \frac{ds}{s} \right]^{p/2} dy \right]^{1/p} < \infty. \]

Then the following holds.

**Theorem.** \(u(x, t) \in H^{\alpha,p}\) if and only if \(u(x, t) = f_\phi \varphi_t(x)\), where \(f \in L^p_{\text{loc}}(\mathbb{R}^n)\) and

\[ \sup_{x,t} \left[ \frac{1}{t^{\gamma + \alpha p}} \int_{\rho(x-y) < t} \left| f(y) - au_{x,t}f \right|^p dy \right]^{1/p} < \infty, \]

where

\[ au_{x,t}f = \frac{1}{|\{z : \rho(z) \leq t\}|} \int_{\rho(x-y) < t} f(y) \, dy. \]

Therefore the spaces of functions \(f(x)\) which arise as traces of functions \(u(x, t)\) in \(H^{\alpha,p}\) are global Lipschitz classes for \(0 < \alpha < 1\) and BMO for \(\alpha = 0\).

**BIBLIOGRAPHY**


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