ON THE CLASSIFICATION OF TAUT SUBMANIFOLDS

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All terminology will be smooth. A submanifold \( K^{2n} \subset M^{2n+2} \) is taut if \( \pi_i(U, \partial U) = 0 \) for \( i \leq n \), where \( U = (M\text{-neighborhood } K) \). Examples are: nonsingular algebraic hypersurfaces in \( CP^n \) (this follows from the Lefschetz theorem on hyperplane sections), simple knots (see [L]), the spines (see [M]). Every codimension-2 homology class contains taut representatives (see [K-M]), and the set of taut submanifolds is closed under connected sum (of pairs) with \( (S^n \times S^n \cong_{\text{standard}} S^{2n+2}) \). Taut submanifolds are "almost canonical" in the sense of [Q], and from this viewpoint it is readily seen that if \( n \geq 3 \), every \( K^{2n} \subset M^{2n+2} \) with \( i \)-connected is concordant to \( K^{2n} \subset M^{2n+2} \) taut.

If \( M^{2n+2} \) is simply connected, the homology groups of \( K^{2n} \), taut, are completely determined by the homology of \( M^{2n+2} \) except for \( B_n(K^{2n}) \). A lower bound on \( B_n(K) \) in terms of \( i_*[K^{2n}] \) and the cohomology ring of \( M^{2n+2} \) has been obtained in [T-W]. In [F1] we have proven Theorem 1, which provides a partial converse to Theorem 2.2 of [T-W] for \( M = CP^{n+1} \), \( n \geq 2 \) odd, and \( i_*[K] \) a prime, \( p \), multiple of the generator of \( H_{2n}(CP^{n+1}; Z) \). Interestingly, if \( p > 3 \), the nonsingular algebraic hypersurfaces \( V \) are not the simplest taut submanifolds in their homology class, but may be decomposed as \( V = K \# l\text{-copies } S^n \times S^n, l > 0 \), for some taut submanifold \( K \).

We do not know if this is true for \( n = 1 \). If it were, there would be surfaces imbedded in \( CP^2 \) with genus smaller than that of the nonsingular algebraic hypersurfaces to which they are homologous. This would contradict Thom’s conjecture.

Statement of Theorem 1. Let \( M^{2n+2} \) be a simply-connected, oriented, smooth \((2n + 2)\)-manifold, \( n \) odd \( > 1 \). Let \( x \in H^2(M^{2n+2}; Z) \) generate a free summand of \( H^2(M^{2n+2}; Z) \). Let \( p \) be any prime. Set

\[ s_{\text{even}} = \max \{ 4, (cosh(p - 2k)x)(sech(px))(L(M))[M] | 0 < k < p \}, \]

\[ s_{\text{odd}} = \max \{ 3, (cosh(p - 2k)x)(sech(px))(L(M))[M] | 0 < k < p \}, \]

where \( L \) is the Hirzebruch polynomial.

For all integers \( h \geq 0 \), there exists a taut submanifold \( K_h \subset M \) with

\[ M \cap px = i_*[K_h], \]

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and
\[ B_n(K_h) = \bar{\tau}_{\text{even}} + 6T_n(M) - 2B_n(M) + B_{n+1}(M) + 2h, \]
if \( B_{n+1}(M) \) is even.
\[ = \bar{\tau}_{\text{odd}} + 6T_n(M) - 2B_n(M) + B_{n+1}(M) + 2h, \]
if \( B_{n+1}(M) \) is odd,

\[ B_n(M) = \text{rank } H_n(M; Z)/\text{Torsion}, \]
\[ T_n(M) = \text{rank } H_n(M) = \text{rank } H_n(M; Z). \]

We now state two theorems, proved in [F2], which indicate to what extent the diffeomorphism class of a taut submanifold is fixed by \( B_n(K) \).

**Theorem 2.** If \( M^{2n+2} \) is a compact, simply connected, smooth \((2n+2)\)-manifold, \( n \) odd \( \geq 3 \), and \( K_0 \xleftarrow{i_0} M^{2n+2} \) and \( K_1 \xrightarrow{i_1} M^{2n+2} \) are \( n \)-connected inclusions of closed submanifolds with \((i_0)_*[K_0] = (i_1)_*[K_1] \in H_2n(M^{2n+2}; Z)\), then if \( B_n(K_0) = B_n(K_1) \), \( K_0 \) is diffeomorphic to \( K_1 \).

**Theorem 3.** Assume \( M^{2n+2} \) is a simply-connected smooth \((2n+2)\)-manifold, \( n \) even, \( \geq 2 \), with \( H_n(M; Z) = 0 \). If \( i_0 \) and \( i_1 \) are as above, then if the intersection pairings on \( H_n(K_0; Z)/\text{Torsion} \) and \( H_n(K_1; Z)/\text{Torsion} \) are isometric, \( K_0 \) is diffeomorphic to \( K_1 \).

If \( M^{2n+2} \), \( n \) odd, \( \geq 3 \), is simply-connected, it follows from Theorem 2 that there is a simplest taut submanifold representing \( i_*[K] \), \( K_0 \), and every other is of the form \( K_l = K_0 \#_l \text{copies } S^n \times S^n \). This, together with a previous remark, yields a complete classification of taut submanifolds in a homotopy \( CP^{n+1} \), \( n \) odd, \( \geq 1 \), representing a prime multiple of the generator of \( H_2n(CP^{n+1}; Z) \).

**References**


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