ON THE CLASSIFICATION OF TAUT SUBMANIFOLDS

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All terminology will be smooth. A submanifold $K^{2n} \subset M^{2n+2}$ is taut if $\pi_i(U, \partial U) = 0$ for $i < n$, where $U = (M$-neighborhood $K)$. Examples are: nonsingular algebraic hypersurfaces in $CP^n$ (this follows from the Lefschetz theorem on hyperplane sections), simple knots (see [L]), the spines (see [M]). Every codimension-2 homology class contains taut representatives (see [K-M]), and the set of taut submanifolds is closed under connected sum (of pairs) with $(S^n \times S^n \subset S^{2n+2})$. Taut submanifolds are “almost canonical” in the sense of [Q], and from this viewpoint it is readily seen that if $n \geq 3$, every $K^{2n} \subset M^{2n+2}$ with $i$ $n$-connected is concordant to $K^{2n} \subset M^{2n+2}$ taut.

If $M^{2n+2}$ is simply connected, the homology groups of $K^{2n}$, taut, are completely determined by the homology of $M^{2n+2}$ except for $B_n(K^{2n})$. A lower bound on $B_n(K)$ in terms of $i_*(K^{2n})$ and the cohomology ring of $M^{2n+2}$ has been obtained in [T-W]. In [F1] we have proven Theorem 1, which provides a partial converse to Theorem 2.2 of [T-W] for $M \cong CP^n+1$, $n > 2$ odd, and $i_*(K)$ a prime, $p$, multiple of the generator of $H_{2n}(CP^n+1; Z)$. Interestingly, if $p > 3$, the nonsingular algebraic hypersurfaces $V$ are not the simplest taut submanifolds in their homology class, but may be decomposed as $V = K \# l$-copies $S^n \times S^n$, $l > 0$, for some taut submanifold $K$.

We do not know if this is true for $n = 1$. If it were, there would be surfaces imbedded in $CP^2$ with genus smaller than that of the nonsingular algebraic hypersurfaces to which they are homologous. This would contradict Thom’s conjecture.

Statement of Theorem 1. Let $M^{2n+2}$ be a simply-connected, oriented, smooth $(2n + 2)$-manifold, $n$ odd $> 1$. Let $x \in H^2(M^{2n+2}; Z)$ generate a free summand of $H^2(M^{2n+2}; Z)$. Let $p$ be any prime. Set

$$ s_{\text{even}} = \max \{4, (\cosh(p - 2k)x)(\text{sech}(px))(L(M))[M] | 0 < k < p \}, $$

$$ s_{\text{odd}} = \max \{3, (\cosh(p - 2k)x)(\text{sech}(px))(L(M))[M] | 0 < k < p \}, $$

where $L$ is the Hirzebruch polynomial.

For all integers $h > 0$, there exists a taut submanifold $K_h \subset M$ with

$$ M \cap px = i_*(K_h), $$

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and
\[
B_n(K_h) = \bar{\tau}_{\text{even}} + 6T_n(M) - 2B_n(M) + B_{n+1}(M) + 2h,
\]
if $B_{n+1}(M)$ is even.

\[
= \bar{\tau}_{\text{odd}} + 6T_n(M) - 2B_n(M) + B_{n+1}(M) + 2h,
\]
if $B_{n+1}(M)$ is odd,

$B_n(M) = \text{rank } H_n(M; Z)/\text{Torsion}$, $T_n(M) = \text{rank } H_n(M) = \text{rank } H_n(M; Z)$.

We now state two theorems, proved in \[F2\], which indicate to what extent the diffeomorphism class of a taut submanifold is fixed by $B_n(K)$.

**Theorem 2.** If $M^{2n+2}$ is a compact, simply connected, smooth $(2n+2)$-manifold, $n$ odd $\geq 3$, and $K_0 \overset{i_0}{\rightarrow} M^{2n+2}$ and $K_1 \overset{i_1}{\rightarrow} M^{2n+2}$ are $n$-connected inclusions of closed submanifolds with $(i_0)_*\{K_0\} = (i_1)_*\{K_1\} \in H_2_n(M^{2n+2}; Z)$, then if $B_n(K_0) = B_n(K_1)$, $K_0$ is diffeomorphic to $K_1$.

**Theorem 3.** Assume $M^{2n+2}$ is a simply-connected smooth $(2n+2)$-manifold, $n$ even, $\geq 2$, with $H_n(M; Z) = 0$. If $i_0$ and $i_1$ are as above, then if the intersection pairings on $H_2_n(K_0; Z)/\text{Torsion}$ and $H_2_n(K_1; Z)/\text{Torsion}$ are isometric, $K_0$ is diffeomorphic to $K_1$.

If $M^{2n+2}$, $n$ odd, $\geq 3$, is simply-connected, it follows from Theorem 2 that there is a simplest taut submanifold representing $i_*\{K\}$, $K_0$, and every other is of the form $K_i = K_0 \#_{l-\text{copies}} S^n \times S^n$. This, together with a previous remark, yields a complete classification of taut submanifolds in a homotopy $CP^{n+1}$, $n$ odd, $\geq 1$, representing a prime multiple of the generator of $H_2_n(CP^{n+1}; Z)$.

**References**


\[F2\] ———, *Uniqueness theorems for taut submanifolds* (to appear).


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