Let $G$ denote the complex symplectic group which may be defined by the equation
\[ G = \left\{ g \in \text{GL}(2k, \mathbb{C}) : gs_kg^t = s_k, \ s_k = \begin{bmatrix} 0 & -I_k \\ I_k & 0 \end{bmatrix} \right\}. \]

In this paper we shall give a simple and concrete realization of a set of representatives of all irreducible holomorphic representations of $G$. This realization, which involves the $G$-module structure of a symmetric algebra of polynomial functions is inspired by the work of B. Kostant [1] and follows the general scheme formulated in [2]. Detailed proofs will appear elsewhere.

1. The symmetric algebra $S(E^*)$. Set $E = \mathbb{C}^n \times 2k$ with $k \geq n \geq 2$; then $G$ acts linearly on $E$ by right multiplication. Let $(\cdot, \cdot)$ denote the skew-symmetric bilinear form on $E$ given by
\[ (X, Y) = \text{trace}(Xs_k Y^t), \quad \forall X, Y \in E. \]

If $X \in E$, let $X^*$ denote the linear form $Y \mapsto (X, Y)$ on $E$. The map $X \mapsto X^*$ establishes an isomorphism between $E$ and its dual $E^*$. Let $S(E^*)$ denote the symmetric algebra of all complex-valued polynomial functions on $E$. The action of $G$ on $E$ induces a representation $R$ of $G$ on $S(E^*)$ defined by
\[ (R(g)p)(X) = p(Xg), \quad \forall p \in S(E^*), \quad \forall X \in E. \]

If $X \in E$, define a differential operator $X^*(D)$ on $S(E^*)$ by setting
\[ (X^*(D)f)(Y) = \{(d/dt)f(Y + tX)\}_{t=0}, \]
for all $f \in S(E^*), t \in \mathbb{R},$ and $X, Y \in E$.

Define $(X_1^* \cdots X_n^*)(D)f = X_1^*(D)((X_2^* \cdots X_n^*)(D)f)$ inductively on $n$. If
\[ [X_1^* \cdots X_l^*(D)] Y_1^* \cdots Y_m^* \]

\[
= \begin{cases} 
0, & \text{if } m < l, \\
\frac{(-1)^l}{(m-l)!} \sum_{\sigma \in S_m} X_1^*(Y_{\sigma(1)}) \cdots X_l^*(Y_{\sigma(l)}) Y_{\sigma(l+1)} \cdots Y_{\sigma(m)}, & \text{if } m \geq l.
\end{cases}
\]

It follows from the above equation and by linearity that the map \( X^* \rightarrow X^*(D) \) extends to an isomorphism \( p \rightarrow p(D) \) between \( S(E^*) \) and the symmetric algebra \( S(E) \) of differential operators on \( E \).

A polynomial \( f \in S(E^*) \) will be called \( G \)-invariant if \( R(g)f = f \), \( \forall g \in G \). A differential operator \( p(D) \in S(E) \) will be called \( G \)-invariant if \( R(g)(p(D)f) = p(D)(R(g)f) \), for all \( g \in G, f \in S(E^*) \). It is then shown that \( p \in S(E^*) \) is \( G \)-invariant if and only if \( p(D) \) is \( G \)-invariant.

Let \( J(E^*) \) (resp. \( J(E) \)) denote the subalgebra of \( S(E^*) \) (resp. of \( S(E) \)) consisting of all \( G \)-invariant polynomials (resp. of all \( G \)-invariant differential operators). Let \( J^+(E^*) \) denote the set of all \( G \)-invariant polynomials without constant terms; \( J^+(E) \) is then defined in a similar fashion.

A polynomial \( f \in S(E^*) \) is said to be \( G \)-harmonic if \( p(D)f = 0 \) for all \( p \in J^+(E^*) \). Let \( H(E^*) \) denote the subspace of \( S(E^*) \) consisting of all \( G \)-harmonic polynomials. Let \( J^+(E^*)S(E^*) \) be the ideal in \( S(E^*) \) generated by \( J^+(E^*) \), and denote by \( V \) the algebraic variety in \( E \) of common zeros of polynomials in the ideal \( J^+(E^*)S(E^*) \). It follows from the theory of polynomial invariants (cf. [3, Chapter VI]) that \( J(E^*) \) is generated by the constant function 1 and \( n(n-1)/2 \) polynomials \( p_{ij} \) defined by

\[
p_{ij}(X) = \sum_{k=1}^{k} (X_{i+k}X_{j+k}^* - X_{i}X_{j+k}^*), \quad 1 \leq i < j \leq n; X = (X_{rs}) \in E.
\]

Moreover, we have \( V = \{ X \in E; Xs \cdot X^t = 0 \} \) and that \( H(E^*) = \{ f \in S(E^*); p_{ij}(D)f = 0, \forall i, j, 1 \leq i < j \leq n \} \). It is then shown that the ideal \( J^+(E^*)S(E^*) \) is prime.

**Theorem 1.1.** The space \( S(E^*) \) is decomposed into a direct sum as \( S(E^*) = J^+(E^*)S(E^*) \oplus H(E^*) \). Moreover, \( S(E^*) = J(E^*) \oplus H(E^*) \) and \( H(E^*) \) is spanned by all polynomials \( (X^*)^m, m = 1, 2, \ldots \), for all \( X \in V \).

**Corollary 1.2.** If \( S(V) \) denotes the ring of functions on \( V \) obtained by restricting elements of \( S(E^*) \) to \( V \), then the restriction mapping \( f \rightarrow f/V \) (\( f \in H(E^*) \)) is a \( G \)-module isomorphism of \( H(E^*) \) onto \( S(V) \).

2. The irreducible holomorphic representations of \( G \). Let \( B \) denote the lower triangular subgroup of \( \text{GL}(n, \mathbb{C}) \) and define a holomorphic character \( \xi = \)
where the $m_i$'s $(1 \leq i \leq n)$ are integers satisfying $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$. A polynomial $f \in S(E^*)$ will be called $\xi$-covariant if $f(bX) = \xi(b)f(X)$, $\forall (b, X) \in B \times E$. Let $H(E, \xi)$ denote the subspace of $H(E^*)$ consisting of all $\xi$-covariant $G$-harmonic polynomials.

**Theorem 2.1.** If $R(\cdot, \xi)$ denotes the representation of $G$ which is obtained by right translation on $H(E, \xi)$ then $R(\cdot, \xi)$ is irreducible and its highest weight is indexed by $(m_1, m_2, \ldots, m_n, 0, \ldots, 0)$ ($k$ factors).

**Proof.** Let

$$C = \left\{ \begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix} \in \text{GL}(2k, \mathbb{C}) : c \text{ diagonal } k \times k \text{ matrix} \right\}$$

and

$$U = \left\{ \begin{bmatrix} u_1 & 0 \\ u_2 & u_1 \end{bmatrix} : u_1^r = (u_1^t)^{-1}, u_1^ru_2^t - u_2u_1^{-1} = 0, \text{ } u_1 \text{ lower triangular unipotent} \right\} ;$$

then $CU$ is a Borel subgroup of $G$. Define a holomorphic character $\xi$ on $CU$ by setting

$$\xi(cu) = c_m^{m_1} \cdots c_n^{m_n}, \forall cu \in CU.$$ 

Let $\text{Hol}(G, \xi)$ denote the space of all $\xi$-covariant holomorphic functions on $G$. Then by the Borel-Weil-Bott theorem the representation $\pi(\cdot, \xi)$ of $G$ which is obtained by right translation on $\text{Hol}(G, \xi)$ is irreducible (see also [4, Chapter XVI]). Let $I = [I_n \ 0] \in E$, then $\text{Orb}(I) = \{ Ig : g \in G \}$ is a dense subset of $V$. Define a map $\Phi$ from $H(E, \xi)$ into $\text{Hol}(G, \xi)$ by the equation $(\Phi f)(g) = f(ig)$, $\forall f \in H(E, \xi), \forall g \in G$. Then it follows from Corollary 1.2 that $\Phi$ is a $G$-module isomorphism. \(\square\)

When $k = n$, the following theorem is an immediate consequence of Theorem 2.1.

**Theorem 2.2.** Suppose that

$$E = \mathbb{C}^{k \times 2k} \ (k \geq 2) \text{ and } \xi = \xi(m_1, m_2, \ldots, m_k);$$

then the representations $R(\cdot, \xi)$ of $G$ on the various spaces $H(E, \xi)$ realize up to equivalence all irreducible holomorphic representations of $G$ when the $m_i$'s $(1 \leq i \leq k)$ are allowed to take all integral values subject to the condition $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$. Moreover, to each representation $R(\cdot, \xi)$ corresponds a highest weight vector $f_\xi \in S(E^*)$ defined by the equation...
\[ f_k(X) = \Delta_{1}^{m-1} \Delta_{2}^{m-2} \Delta_{3}^{m-3} \cdots \Delta_{k-1}^{m-k} \Delta_{k}^{m} (X), \quad \forall X \in E \]

where the \( \Delta_i(X) \) \((1 \leq i \leq k)\) are the principal minors of \( X \).

REFERENCES


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