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NOVIKOV'S EXT^2 AND THE NONTRIVIALITY OF THE GAMMA FAMILY

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Larry Smith [7] defined and detected elements β_t in the p-primary component of the stable homotopy of the sphere for $t > 0$ and $p \geqslant 5$. In the same manner, Toda's construction [11] gives elements γ_t for $t > 0$ and $p \geqslant 7$. We have the following results which are a consequence of our computation of the second line of the Novikov spectral sequence for a sphere at an odd prime.

THEOREM 1. (a) *p does not divide* $\beta_t \in \pi^S_{2(p^2-1)t-2(p-1)-2}(S^0)$ *for* $p \geqslant 5$, $t > 0$.

(b) $0 \neq \gamma_t \in \pi^S_{2(p^3-1)t-2(p^2-1)-2(p-1)-3}(S^0)$ *for* $p \geqslant 7, t > 0$.

(c) $\alpha_1\beta_t \neq 0$ *for* $t \neq 0$ *or* -1 mod p, $p \geqslant 5$.

Partial results on the nontriviality of γ_t have been obtained by Thomas and Zahler [10], [9], Oka and Toda [6], Johnson, Miller, Wilson, and Zahler [2], and Ravenel (unpublished).

These infinite families can be studied most conveniently by means of the Novikov spectral sequence

$$E_2^{**} = \mathrm{Ext}^{**}_{BP_*BP}(BP_*, BP_*(X)) \Rightarrow \pi_*(X)_{(p)}$$

for a space X [1]. $BP_*(\ )$ is the Brown-Peterson homology theory [1], and

$$BP_* = BP_*(S^0) = \mathbf{Z}_{(p)}[v_1, v_2, \ldots], \qquad |v_i| = 2(p^i - 1).$$

Let I_n denote the invariant ideal $(p, v_1, \ldots, v_{n-1}) \subset BP_*$; $I_0 = (0)$. For a BP_*BP comodule M let H^*M denote $\mathrm{Ext}^{**}_{BP_*BP}(BP_*, M)$. By a theorem of Landweber [3] we have for $n > 0$

$$H^0BP_*/I_n = \mathbf{F}_p[v_n].$$

Let δ_n: $H^iBP_*/I_{n+1} \longrightarrow H^{i+1}BP_*/I_n$ be the connecting homomorphism in the long exact sequence associated with

$$0 \longrightarrow BP_*/I_n \xrightarrow{v_n} BP_*/I_n \longrightarrow BP_*/I_{n+1} \longrightarrow 0.$$

It is folklore (see [2]) that if $p \geqslant 7$, $t > 0$, and $0 \neq \delta_0\delta_1\delta_2(v_3^t) \in H^3BP_*$, then this class survives to γ_t and $\gamma_t \neq 0$. Our proof of Theorem 1 involves an analysis

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of the groups involved in computing $\delta_0\delta_1\delta_2(v_3^t)$. For $p > 2$, the necessary H^1BP/I_n were computed by two of the authors and announced in [4]. A corollary of this result is that $0 \neq \delta_1\delta_2(v_3^t) \in H^2BP_*/(p)$ (see [2]). Hence it remains to study the exact sequence

$$H^2BP_* \xrightarrow{\rho_0} H^2BP_*/(p) \xrightarrow{\delta_0} H^3BP_*.$$

The proof that $\delta_1\delta_2(v_3^t) \notin \operatorname{Im} \rho_0$ rests on a complete calculation of H^2BP_*. We now describe this group.

First define a sequence of elements $x_i \in v_2^{-1}BP_*$ by

$$\begin{aligned} x_0 &= v_2, \\ x_1 &= v_2^p - v_1^p v_2^{-1} v_3, \\ x_2 &= x_1^p - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3, \\ x_n &= x_{n-1}^p - 2 v_1^{(p+1)(p^{n-1}-1)} v_2^{p^n-p^{n-1}+1} \quad \text{for } n \geq 3. \end{aligned}$$

Also let $a_0 = 1$ and $a_j = p^j + p^{j-1} - 1$ for $j \geq 1$.

Now $BP_*/(p^{i+1}, v_1^{mp^i})$ is a BP_*BP-comodule for $m > 0$, and we have

LEMMA 2. *$x_{k+i}^a \in H^0BP_*/(p^{i+1}, v_1^{mp^i})$ for*

$$0 < m \leq \begin{cases} p^{k-i} & \text{if } i = 0, a = 1, \\ a_{k-i} & \text{otherwise.} \end{cases}$$

Let

$$H^0BP_*/(p^{i+1}, v_1^{mp^i}) \xrightarrow{\delta''} H^1BP_*/(p^{i+1}) \xrightarrow{\delta'} H^2BP_*$$

denote the connecting homomorphisms associated with the obvious short exact sequences. Let

$$\beta_{ap^{k+i}/(mp^i, i+1)} = \delta'\delta''(x_{k+i}^a)$$

for a, i, k, m as in Lemma 2, and abbreviate $\beta_{n/(i,1)} = \beta_{n/(i)}$, $\beta_{n/(1)} = \beta_n$. Then our main result is

THEOREM 3. *Let $p \geq 3$. The graded $\mathbf{Z}_{(p)}$-module H^2BP_* is the direct sum of cyclic modules generated by $\beta_{ap^{2i+j}/(mp^i, i+1)}$, of order p^{i+1} for $k \geq i \geq 0$, $(a, p) = 1$, $a > 0$, with m as in Lemma 2, but $m > a_{j-1}$ if $p \mid m$, $k = i + j$.*

REMARK 4. The lowest dimensional element of order p^{i+1} occurs when $k = i$ and $a = m = 1$ in dimension $2(p^2 - 1)p^{2i} - 2(p - 1)p^i$.

For $p \geq 5$, the stable homotopy element $\epsilon_r(t)$, $t > 0$, $0 < r \leq p - 1$, of L. Smith [8] is represented by $\beta_{tp/(p-r)}$ and Oka's $\rho'_{pt,r}$ [5], $t > 0$, $0 < r \leq 2(p - 1)$ by $\beta_{p^2t/(2p-1-r)}$. Smith's element β_t is represented by our β_t. Thus none of these elements is divisible by p^2, and of them only $\rho'_{pt,p-1}$ can possibly be divisible by p.

Our techniques lead to much new information about products in H^3BP_*, such as 1(c). This will appear elsewhere.

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