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JOINT SPECTRUM IN THE CALKIN ALGEBRA

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For a nice discussion pertaining to the essential spectrum of a single operator (bounded linear transformation) in a complex separable infinite dimensional Hilbert space $\mathcal{H}$, the reader is referred to Fillmore, Stampfli and Williams [4]. The purpose of this note is to announce analogous results concerning the joint essential spectra of n-tuples of operators in $\mathcal{H}$.

Joint essential spectrum. In the sequel $\mathcal{L}(\mathcal{H})$ denotes the algebra of all operators on $\mathcal{H}$ and $\mathcal{K}$ denotes the ideal of compact operators on $\mathcal{H}$. Let ν be the canonical homomorphism from $\mathcal{L}(\mathcal{H})$ onto the *Calkin algebra* $\mathcal{L}(\mathcal{H})/\mathcal{K} = \mathcal{C}$. If $A = (A_1, \ldots, A_n)$ is an n-tuple of operators on $\mathcal{H}$, then we write $\nu(A_j) = a_j$, the coset containing A_j for each j, $1 \leqslant j \leqslant n$, and $a = (a_1, \ldots, a_n)$.

The *joint essential spectrum* of an n-tuple of operators A denoted by $\sigma_e(A)$ is defined to be the *joint spectrum* $\sigma(a)$ of a.

Here $\sigma(a) = \sigma^l(a) \cup \sigma^r(a)$, where the *left* (*right*) *joint spectrum* $\sigma^l(a)$ ($\sigma^r(a)$) is defined as the set of all $z = (z_1, \ldots, z_n)$ in $\mathbf{C}^n$ (n-fold Cartesian product of the set of all complex numbers $\mathbf{C}$) such that $\{a_j - z_j\}_{1 \leqslant j \leqslant n}$ generates a proper left (right) ideal in the Calkin algebra $\mathcal{C}$. For this notion of joint spectrum, the reader may consult [1] and [5]. We call the set $\sigma^l(a)$ ($\sigma^r(a)$) as the *left* (*right*) *joint essential spectrum* and denote it by $\sigma_e^l(A)$ ($\sigma_e^r(A)$). Clearly, $\sigma_e^l(A) \subseteq \sigma^l(A)$, $\sigma_e^r(A) \subseteq \sigma^r(A)$; and hence $\sigma_e(A) \subseteq \sigma(A)$. Further, if $A = (A_1, \ldots, A_n)$ is an n-tuple of *essentially commuting* (commuting modulo the compacts) operators, then $\sigma_e(A)$ is a nonempty compact subset of $\mathbf{C}^n$.

The following theorem describes the relationship between the joint spectrum and the joint essential spectrum of an n-tuple of operators.

THEOREM 1. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of operators on $\mathcal{H}$. Then $\sigma(A) = \sigma_e(A) \cup \sigma_p(A) \cup \sigma_p(A^*)^*$, where $A^* = (A_1^*, \ldots, A_n^*)$ and star on the right represents complex conjugates.*

A point $z = (z_1, \ldots, z_n)$ of $\mathbf{C}^n$ is in $\sigma_p(A)$ (the *joint eigenvalue* of A) if

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and only if there exists a nonzero vector f in H such that $(A_j - z_j)f = 0$ for each j, $1 \leqslant j \leqslant n$ (consult [3]).

COROLLARY 1. *Let* $A = (A_1, \ldots, A_n)$ *be as given above. Then*:

(a) $\sigma^l(A)$ *consists of* $\sigma_e^l(A)$ *together with the joint eigenvalues of finite multiplicity.*

(b) $\sigma^r(A)$ *consists of* $\sigma_e^r(A)$ *together with the set of all* $z = (z_1, \ldots, z_n)$ *in* $\mathbf{C}^n$ *such that* z^* *is a joint eigenvalue of finite multiplicity of* A^*.

The next theorem characterizes the joint essential spectrum of special operators.

THEOREM 2. *Let* $A = (A_1, \ldots, A_n)$ *be an n-tuple of essentially hyponormal operators* $(a_j a_j^* \leqslant a_j^* a_j, 1 \leqslant j \leqslant n)$. *Then* $\sigma_e(A) = \sigma_e^r(A)$.

COROLLARY 2 [2, LEMMA 2.1]. *Let* $A = (A_1, \ldots, A_n)$ *be an n-tuple of essentially normal* ($A_j^*A_j - A_jA_j^*$ *is compact for each j,* $1 \leqslant j \leqslant n$) *operators. Then* $\sigma_e(A) = \sigma_e^l(A)$.

Joint eigenvalues in the Calkin algebra. It is known that if $b \in \mathcal{C}$ and $z \in \sigma(b)$, then there is a projection $p \neq 0$ such that $bp = zp$ or $pb = zp$ [4]. The following theorem is an extension of this result to n-tuples of elements in $\mathcal{C}$.

THEOREM 3. *Let* $a = (a_1, \ldots, a_n)$ *be an n-tuple of elements in the Calkin algebra* $\mathcal{C}$ *and* $z = (z_1, \ldots, z_n) \in \sigma(a)$. *Then there is a projection* $p \neq 0$ *such that either* $a_j p = z_j p$ *for all j,* $1 \leqslant j \leqslant n$, *or* $pa_j = z_j p$ *for all j,* $1 \leqslant j \leqslant n$.

COROLLARY 3. *Let* $A = (A_1, \ldots, A_n)$ *be an n-tuple of essentially commuting operators. Then there are orthogonal projections P and Q of infinite rank and nullity and a point* $z = (z_1, \ldots, z_n)$ *of* $\mathbf{C}^n$ *such that* $(A_j - z_j)P$ *is compact for all j,* $1 \leqslant j \leqslant n$, *and* $Q(A_j - z_j)$ *is compact for all j,* $1 \leqslant j \leqslant n$.

COROLLARY 4. *Let* $A = (A_1, \ldots, A_n)$ *be an n-tuple of essentially commuting operators. Then the operators* $A_1, \ldots, A_n$ *have a common invariant subspace "modulo the compacts".*

THEOREM 4. *Let* $a = (a_1, \ldots, a_n)$ *be an n-tuple of hyponormal elements in the Calkin algebra* $\mathcal{C}$. *Then*:

(a) $z = (z_1, \ldots, z_n) \in \sigma(a)$ *if and only if there is a projection* $p \neq 0$ *such that* $a_j^* p = z_j^* p$ *for all j,* $1 \leqslant j \leqslant n$.

(b) *If p is a projection such that* $a_j p = z_j p$, $1 \leqslant j \leqslant n$, *then* $a_j^* p = z_j^* p$, $1 \leqslant j \leqslant n$.

The essential key to most of the results above is the following:

THEOREM 5. *The following statements are equivalent*:

(1) $0 = (0, 0, \ldots, 0) \in \sigma_e^l(A_1, \ldots, A_n)$.

(2) $0 \in \sigma_e(\Sigma_{j=1}^n A_j^* A_j)$.

(3) *There exists an orthogonal sequence* $\{e_k\}$ *such that* $\|A_j e_k\| \longrightarrow 0$ *as* $k \longrightarrow \infty$, *for each* j, $1 \leqslant j \leqslant n$.

(4) *There exists an infinite dimensional projection* P *such that* $A_j P$ *is compact for each* j, $1 \leqslant j \leqslant n$.

(5) $\Sigma_{j=1}^n A_j^* A_j$ *is not Fredholm.*

(6) $0 \in \sigma^l(a_1, \ldots, a_n)$.

(7) $0 \in \sigma(\Sigma_{j=1}^n a_j^* a_j)$.

REMARK. Most of the results above can be extended to sequences $\{A_n\}$ of operators with very little modifications in the proofs. However, for brevity, we have chosen to discuss them for n-tuples of operators in H.

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