CLASSIFICATION OF AUTOMORPHISMS OF HYPERFINITE FACTORS OF TYPE II₁ AND IIₙ AND APPLICATION TO TYPE III FACTORS

BY A. CONNES

Communicated by I. M. Singer, April 10, 1975

ABSTRACT. For each integer \( p = 0, 1, 2, \ldots \) and complex number \( \gamma \), \( \gamma^p = 1 \) (\( \gamma = 1 \) for \( p = 0 \)) we define an automorphism \( s^p_\gamma \) of the hyperfinite factor of type II₁, \( R \). For any automorphism \( \alpha \) of \( R \) there is a unique couple \( (p, \gamma) \) and a unitary \( v \in R \) such that \( \alpha \) is conjugate to \( \text{Ad} \, v \circ s^\gamma_p \). Let \( R_{0,1} \) be the tensor product of \( R \) by a \( I_{\infty} \) factor. There is, up to conjugacy, only one automorphism \( \theta_\lambda \) of \( R_{0,1} \) such that \( \theta_\lambda \) multiplies the trace by \( \lambda \), provided \( \lambda \neq 1 \).

Introduction. The classification of type III factors that we proposed in [2] relates isomorphism classes of type III₁ factors, \( \lambda \in ]0, 1[ \) with outer conjugacy classes of automorphisms of factors of type IIₙ. An obvious criticism to the value of such a relation is then the following: Is it possible to classify automorphisms even for the simplest factor of type IIₙ, namely \( R_{0,1} \), the hyperfinite II₁, by a \( I_{\infty} \) factor. We answer this question in this paper, showing that for any \( \lambda \in ]0, 1[ \) there is only one automorphism, up to conjugacy, of \( R_{0,1} \) which multiplies the trace by \( \lambda \). The proof of this fact relies on the classification of automorphisms of the hyperfinite factor \( R \) (see Theorem 1) which in turn uses mainly the analogy between classical ergodic theory and ergodic theory on a nonabelian von Neumann algebra.

Automorphisms of the hyperfinite factor of type II₁. Recall that if \( M \) is a factor and \( \theta \in \text{Aut} \, M \), one defines the outer period \( p_0(\theta) \) as the period of \( \theta \) modulo inner automorphisms (i.e. \( \theta^k \in \text{Int} \, M \Leftrightarrow k \in p_0(\theta)\mathbb{Z} \)). Also the obstruction of \( \theta \), noted \( \gamma(\theta) \), is the root of unity in \( \mathbb{C} \) such that \( \theta^{p_0} = \text{Ad} \, v, v \text{ unitary in } M \Rightarrow \theta(v) = \gamma v \). Finally \( \alpha \) and \( \beta \in \text{Aut} \, M \) are outer conjugate iff \( \beta \) is conjugate to the product of \( \alpha \) by an inner automorphism.

THEOREM 1. Two automorphisms \( \alpha, \beta \) of \( R \) are outer conjugate if and only if \( p_0(\alpha) = p_0(\beta) \) and \( \gamma(\alpha) = \gamma(\beta) \).

In particular, any two aperiodic automorphisms \( \alpha, \beta \) of \( R \) are outer conjugate. This relies on an analogue of Rohklin's theorem. In the case \( p_0(\alpha) \neq 0 \) the proof uses the tensor product as a group structure on the set \( Br(Z/p, R) \) of

outer conjugacy classes of $\alpha$'s with $p_0(\alpha) = p$ (see [5]). For $p \neq 0$ and $\gamma \in C$, $\gamma^p = 1$, there is, up to conjugacy, only one automorphism $s_p^\gamma$ of $R$ with period $p$ order $\gamma$ and invariants $p$, $\gamma$. This automorphism $s_p^\gamma$ has been described in [4], [5]. For $p = 0$ we let $s_0$ be the bilateral shift on $R$ when $R$ is written $R = \bigotimes_{\nu \in \mathbb{Z}}(R_1)_{\nu}$ with $R_1$ isomorphic to $R$.

Theorem 1 means that up to conjugacy any automorphism of $R$ is the product of an $s_p^\gamma$ by an inner automorphism.

**Corollary 2.** The group $\text{Out } R = \text{Aut } R/\text{Int } R$ has no nontrivial normal subgroup.

In particular the center of $\text{Out } R$ is trivial, unlike for any type III factors [2, 1.2.8b].

Corollary 2 means that $R$ cannot break in a significant and invariant way into simpler objects.

**Corollary 3.** Let $N$ be a finite von Neumann algebra generated by a hyperfinite von Neumann subalgebra $P$ and a unitary $U$, $UPU^* = P$, then $N$ is hyperfinite.

One shows, using Theorem 1, that any automorphism $\alpha$ of $P$ generates the same full group [2, 1.5.4] as an automorphism $\beta$ such that for some increasing sequence of finite dimensional subalgebras $K_\nu$, $\nu \in N$, of $P$ one has $\beta(K_\nu) = K_\nu$ for all $\nu$, and $\bigcup_{\nu = 1}^{\infty} K_\nu$ dense in $P$. Then one replaces $U, \text{Ad } U/P = \alpha$ by a unitary $V \in N$ such that $\text{Ad } V/P = \beta$.

With Corollary 3 one can then prove a result, due to Golodets when $N$ is properly infinite (see [6]).

**Corollary 4.** Let $P$ be a hyperfinite von Neumann algebra and $G$ a solvable group of unitaries in $L(H)$ such that $uPu^* = P$ for all $u \in G$; then $(P \cup G)''$ is hyperfinite.

In particular, any representation of a solvable group generates a hyperfinite von Neumann algebra.

**Automorphisms of the known hyperfinite factor of type $\text{II}_{\infty}$.** Let $N$ be a factor of type $\text{II}_{\infty}$, $\tau$ a faithful semifinite normal trace on $N$, $\theta \in \text{Aut } N$; then we call the unique $\lambda \in R_+^*$ such that $\tau \circ \theta = \lambda \tau$ the module of $\theta$: $\lambda = \text{mod } \theta$.

**Theorem 5.** Let $R_{0,1} = R \otimes L(H)$ be the known hyperfinite factor of type $\text{II}_{\infty}$. For any $\lambda \in R_+^*$, $\lambda \neq 1$, there is up to conjugacy only one $\theta \in \text{Aut } R_{0,1}$ with module equal to $\lambda$.

Also, two automorphisms $\alpha, \beta$ or $R_{0,1}$ with module equal to 1 are outer conjugate iff they have the same outer period and the same obstruction.

**Corollary 6.** An automorphism $\theta \in \text{Aut } R_{0,1}$ is a commutator $\theta = \text{null}$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
αα⁻¹β⁻¹ for α, β ∈ Aut R₀,₁ if and only if its module is equal to 1.

**Corollary 7.** Let M be a factor of type IIIₜ and M = W*(θ, N) its discrete decomposition [2, Theorem 4.4.1]. Then M is isomorphic to Powers factor Rₜ iff N is isomorphic to R₀,₁.

**BIBLIOGRAPHY**


DEPARTMENT OF MATHEMATICS, QUEEN'S UNIVERSITY, KINGSTON, ONTARIO, CANADA