THE STRUCTURE OF SINGULARITIES IN AREA-RELATED VARIATIONAL PROBLEMS WITH CONSTRAINTS

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This is a research announcement of results whose full details and proofs have been submitted for publication elsewhere. We provide a complete description, both combinatorial and differential, of the local structure of singularities in a large class of two-dimensional surfaces in $\mathbb{R}^3$, those which are $(M, e, \delta)$ minimal [TJ1] and those which are $(F, e, \delta)$ minimal for a Hölder continuous ellipsoidal integrand $F$ [TJ2]. Such surfaces include mathematical models for compound soap bubbles [AF1], [AF2] and soap films, thereby settling a problem which has been studied for well over a century (a very general formulation of Plateau's Problem); in general, $(M, e, \delta)$ and $(F, e, \delta)$ minimal surfaces arise as solutions to geometric variational problems with constraints.

$(M, e, \delta)$ and $(F, e, \delta)$ minimal surfaces were defined, shown to exist, and proven to be regular almost everywhere in [AF2] (see [AF1] for a brief description). We define $Y \subset \mathbb{R}^3$ as the union of the half disk $\{x \in \mathbb{R}^3: x_1^2 + x_2^2 < 1, x_3 \geq 0, x_3 = 0\}$ with its rotations by $120^\circ$ and $240^\circ$ about the $x_1$ axis, and define $T \subset \mathbb{R}^3$ as $C \cap \{x: |x| < 1\}$, where $C$ is the central cone over the one-skeleton of the regular tetrahedron centered at the origin and containing as vertices the points $(3, 0, 0)$ and $(-1, 2\sqrt{2}, 0)$. Varifold tangents are defined in [AW 3.4] and a tangent cone is defined to be the support of a varifold tangent.

The major result of [TJ1] is the following.

**Theorem.** Suppose $S$ is $(M, e, \delta)$ minimal with respect to some closed set $B$, where $e(r) = Cr^\alpha$ for some $C < \infty$ and $\alpha > 0$. Then

1. there exists a unique tangent cone, denoted $\text{Tan}(S, p)$, to $S$ at each point $p$ in $S$,
2. $R(S) = \{p \in S: \text{Tan}(S, p) \text{ is a disk}\}$ is a two-dimensional Hölder continuously differentiable submanifold of $\mathbb{R}^3$, with $H^2(R(S)) = H^2(S)$ [AF1], [AF2] (here $H^2$ denotes (Hausdorff) two-dimensional area),
3. $\sigma_Y(S) = \{p \in S: \text{Tan}(S, p) = \theta Y \text{ for some } \theta \text{ in } O(3), \text{ the group of}


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orthogonal rotations of \( R^3 \) is a one-dimensional Hölder continuously differentiable submanifold of \( R^3 \), and for each \( p \) in \( \sigma_Y(S) \) there exists a neighborhood \( N \) of \( p \) and a Hölder continuously differentiable diffeomorphism \( f: R^3 \rightarrow R^3 \) such that \( f(S \cap N) = Y \),

\[
\sigma_Y(S) = \{ p \in S: \operatorname{Tan}(S, p) = \theta T \text{ for some } \theta \text{ in } O(3) \} \text{ consists of isolated points, and for each } p \text{ in } \sigma_Y(S) \text{ there is a neighborhood } N \text{ of } p \text{ and a Hölder continuously differentiable diffeomorphism } f: R^3 \rightarrow R^3 \text{ such that } f(S \cap N) = T,
\]

\[
S = R(S) \cup \sigma_Y(T) \cup \sigma_Y(S).
\]

The proof uses the methods of geometric measure theory, in particular those developed in [AF2] and [TJ3], and can be divided into three broad regions: a tangent cone analysis, the proof of an “epiperimetric inequality” similar to those of [R] and [TJ3] but in derivative form, and the derivation of the differential structure from this inequality. Contained in the first region is a proof that the only two-dimensional area minimizing cones in \( R^3 \) are (up to rotations) the disk \( Y \) and \( T \); Lamarle [L] in 1864 attempted to do this, but his analysis was partly in error.

Recall that an integrand is a continuous function \( F: R^3 \times G(3, 2) \rightarrow R^+ \), where \( G(3, 2) \) denotes the Grassmannian of unoriented two-plane directions in \( R^3 \), and that at each point \( p \) in \( R^3 \) there is associated to \( F \) the constant coefficient integrand \( F^p: G(3, 2) \rightarrow R^+ \) given by \( F^p(\pi) = F(p, \pi) \) for every \( \pi \) in \( G(3, 2) \) [AF2]. \( F^p \) may be regarded as a function on the unit ball in \( \bigwedge^2 R^3 \), the space of 2-vectors of \( R^3 \), by defining \( F^p(v) = F^p(\pi) \), where \( v \) is any 2-vector of length 1 and \( \pi \) is the unoriented two-plane naturally associated to \( v \) [Fl, 6.1].

We define \( F \) to be ellipsoidal if for each \( p \) in \( R^3 \) the positively homogeneous function of degree one on \( \bigwedge^2 R^3 \) which extends \( F^p \) is a norm induced by an inner product on \( \bigwedge^2 R^3 \). Equivalently, for every \( p \) in \( R^3 \) there exists a nonsingular linear map \( L_p: R^3 \rightarrow R^3 \) such that \( F^p(v) = |\bigwedge^2 L_p(\nu)| \) for every 2-vector \( v \) of length 1; thus for every \( (H^2, 2) \)-rectifiable set \( S \), \( F^p(S) = H^2( L_p(S)) \).

The major result of [TJ2] is then that if \( S \) is \((F, \epsilon, \delta)\) minimal with respect to some closed set \( B \) for a Hölder continuous ellipsoidal integrand \( F \) (and \( \epsilon(r) = Cr^\alpha \) for some \( C < \infty \) and \( \alpha > 0 \) ), the conclusions of the above theorem hold, except that “\( \theta Y \)” is replaced by “\( L_p^{-1}(\theta Y) \)” and “\( \theta T \)” by “\( L_p^{-1}(\theta T) \)”. This implies in particular that area minimizing surfaces (more generally, \((M, \epsilon, \delta)\) minimal surfaces) on three-dimensional Hölder continuously differentiable manifolds have the structure described in the theorem; such a result could be derived directly from [TJ1] only if the manifold were at least \( C^2 \).

A converse of these results is easy to prove, i.e., if a compact surface has the structure of (1)–(5) of the theorem [respectively, has that structure up to a linear map at each point], then the surface is \((M, \epsilon, \delta)\) minimal [respectively, \((F, \epsilon, \delta)\) minimal for some Hölder continuous ellipsoidal integrand \( F \)] for some \( \delta > 0 \) and \( \epsilon(r) = Cr^\alpha \), some \( C < \infty \) and \( \alpha > 0 \).
REFERENCES


[TJ2] ———, The structure of singularities in solutions to quadratic variational problems with constraints in \( \mathbb{R}^3 \) (preprint).


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