AN $L^1$-SPACE FOR BOOLEAN ALGEBRAS
AND SEMIREFLEXIVITY OF $L^\infty(X, \Sigma, \mu)$

BY DENNIS SENTILLES

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This note indicates how one can use the ideas of strict topologies on spaces of continuous functions to, at a single stroke, obtain an extended construction of $L^1$-spaces without reference to measure, obtain ordinary $L^1(X, \Sigma, \mu)$-spaces as the natural dual of $L^\infty(X, \Sigma, \mu)$ and obtain a view of the dual pairing $(L^\infty, L^1)$ that is very much like that of $(C, M)$, where $C$ is a space of bounded continuous functions and $M$ a space of bounded Baire or Borel measures.

Earlier results, [1] and [4], suggest this development. In [1], Buck shows that $M$, the compact regular Borel measures on locally compact $X$, results as $(C(\mathcal{O}, P))'$, where $\mathcal{O}$ is the topology on $C(X)$ defined by the seminorms $\|f\|_\xi = \sup\{ |f(x)| \xi(x)| : x \in X \}$, with $\xi \in C$ vanishing at $\infty$. In [4], this writer showed how $\beta$-methods extend to completely regular $X$, with $\xi \equiv 0$ over compact sets, or zero sets, in $\beta X \setminus X$. For $X = \{1, 2, \ldots\}$, $L^\infty = C$ and $L^1 = M$, and by [1], $(L^\infty, \beta)' = L^1$. By choosing $\xi \equiv 0$ over certain closed nowhere dense subsets of the appropriate Stone space, we show herein that this result is more than the small coincidence formally expected.

2. The space $L^1(\mathcal{A})$. Let $\mathcal{A}$ be a Boolean algebra [6] and let $\mathcal{S}$ be its Stone space with $\eta(a) \subset \mathcal{S}$ denoting the compact-open set corresponding to $a \in \mathcal{A}$.

We define an indicator function on $\mathcal{A}$, $\chi: \mathcal{A} \rightarrow C(\mathcal{S})$, by $\chi(a) = \chi_{\eta(a)}$ and let $L^\infty(\mathcal{A})$ be the closed linear span of $\chi(\mathcal{A})$ in $C(\mathcal{S})$ in the $\| \| = \text{uniform convergence on } \mathcal{S}$ topology on $C(\mathcal{S})$. In fact, $L^\infty(\mathcal{A}) = C(\mathcal{S})$.

For each increasing sequence $a_n \in \mathcal{A}$ with $a = \sup a_n$ (i.e., $a_n \downarrow a$), let $Q = \eta(a) \cup \bigcup_{n=1}^\infty \eta(a_n)$ and define the $\beta_Q$ topology on $L^\infty = L^\infty(\mathcal{A})$ by the seminorms $\|f\|_\xi$ for $f \in L^\infty$ where $\xi \in C(\mathcal{S})$ and $\xi \equiv 0$ on $Q$. Let $\beta$ be the inductive limit topology over all such $Q$. We remark that $\beta$ may be neither Hausdorff, nor finer than pointwise convergence and is the $\| \|$-topology iff all increasing $\{a_n\}$ with a supremum in $\mathcal{A}$ are finite.

We now define $L^1(\mathcal{A})$ by $L^1(\mathcal{A}) = (L^\infty(\mathcal{A}), \beta)'$, the $\beta$-dual of $L^\infty(\mathcal{A})$. It is possible that $L^1(\mathcal{A}) = \{0\}$ ([6, p. 65] and (2) below).
The crucial result is

**Theorem 1.** \( \chi: \mathcal{A} \to (L^\infty(\mathcal{A}), \beta) \) is a vector measure.

**Proof.** If \( a_n \to a \) and \( W \) is a \( \beta \)-neighborhood of \( 0 \), choose \( \xi \equiv 0 \) on \( Q \) so that \( \|f\|_\xi \leq \epsilon \) puts \( f \in W \). If \( n_0 \) is chosen so that \( \mathcal{S} \eta(a) \cup \eta(a_{n_0}) \supset \{ x : |\xi(x)| \geq \epsilon/2 \} \), then \( \|\chi(a_n) - \chi(a)\|_\xi \leq \epsilon \) for \( n \geq n_0 \) and we are done.

Consequently,

**Theorem 2.** The equality \( \mu(a) = (\hat{\mu} \circ \chi)(a) \) defines a 1-1, onto correspondence between the positive elements \( \hat{\mu} \in L^1(\mathcal{A}) \) and the finite-valued positive measures on \( \mathcal{A} \).

**Proof.** \( \hat{\mu} \circ \chi \) is obviously a measure on \( \mathcal{A} \). Conversely, if \( \mu \) is given,

\[
\phi(\sum_{i=1}^{n} \alpha_i \chi(a_i)) = \sum \alpha_i \mu(a_i)
\]

extends uniquely to a bounded functional on \( L^\infty \) which is then \( \beta \)-continuous because \( \mu \) is a measure.

3. **Spaces** \( L^\infty(X, \Sigma, \mu) \). Let \( \bar{\mathcal{A}} \) be the Boolean algebra \( \Sigma/\mu^{-1}(0) \) under \( \cap, \Delta \), where \( \bar{\mu} \) is \( \sigma \)-finite. Let \( [\cdot] \) denote equivalence classes in \( L^\infty(\bar{\mu}) \) or \( \mathcal{A} \) alternatively. Define (e.g. [6, p. 155]) \( \theta: L^\infty(X, \Sigma, \bar{\mu}) \to L^\infty(\mathcal{A}) \) by \( \theta[E] = \chi[E] \), extended linearly and by uniform closure to all of \( L^\infty(X) \); \( \theta \) is an \( \|\|_\infty - \|\| \) isometry onto. Let \( \beta_\infty \) be the weak topology on \( L^\infty(\bar{\mu}) \) making \( \theta \) \( \beta \) continuous into \( L^\infty(\mathcal{A}) \); \( \theta \) is a \( \beta_\infty - \beta \) homeomorphism.

**Theorem 3.** \( (L^\infty(\bar{\mu}), \beta_\infty)' = L^1(X, \Sigma, \bar{\mu}) = \theta'(L^1(\mathcal{A})) \).

The proof depends on the fact that \( \bar{\nu} \in L^\prime(\bar{\mu}) \) allows \( \nu[E] = \bar{\nu}(E) \) to be a well-defined measure on \( \bar{\mathcal{A}} \).

**Theorem 4.** \( \beta_\infty \) is the finest locally convex topology on \( L^\infty(\bar{\mu}) \) with dual \( L^1(\bar{\mu}) \), and, moreover, on \( \|\|_\infty \)-bounded sets, \( \beta_\infty \) agrees with, and is the finest locally convex topology on \( L^\infty(\bar{\mu}) \) to so agree with, convergence in \( \bar{\mu} \)-measure. Consequently, the \( \beta_\infty \)-continuity of linear maps is determined on such sets by continuity under convergence in \( \bar{\mu} \)-measure.

The proof of this result is most easily obtained through Theorem 6 below.

4. **Further results.** Among other results we select two which seem to most justify our constructions above. For \( \hat{\nu} \in L^1(\mathcal{A}) \) let \( \|\hat{\nu}\| = \sup \{ |\hat{\nu}(f)| : f \in L^\infty, \|f\| \leq 1 \} \). For \( f \in L^\infty \), let \( \phi_f \in (L^1(\mathcal{A}), \|\|)' \) by \( \phi_f(\hat{\nu}) = \hat{\nu}(f) \). By \( (L^1(\mathcal{A}), \|\|)' = L^\infty(\mathcal{A}) \) we mean that the map \( f \to \phi_f \) is an isometry onto \( (L^1(\mathcal{A}), \|\|)' \). This map is 1-1 iff \( \beta \) is \( T_2 \).

**Theorem 5.** If \( \mathcal{A} \) is \( \sigma \)-complete and satisfies the countable chain condition and \( \beta \) is \( T_2 \), then \( (L^1(\mathcal{A}), \|\|)' = L^\infty(\mathcal{A}) \).

The proof depends on the fact that \( L^1(\mathcal{A}) \) coincides with Dixmier's normal measures on \( S \) under these hypotheses. The converse may not generally hold; for example if \( \mathcal{A} = 2^{[0,1]} \) and we assume that no subset of \([0,1]\) has cardinal...
of measure zero. Indeed for algebras without the countable chain condition, one
should go to the topology $\bar{\beta}$ defined by sets $Q = \eta(\sup B) \setminus \bigcup \eta(b)$ where $B \subseteq A$, 
whereupon this theory begins to meet that of [3].

**Theorem 6 (Dunford-Pettis).** For a bounded subset $H$ of $L^1(A)$, these
are equivalent:

1. $H$ is weak* (i.e., $\sigma(L^1, L^\infty)$) countably compact,
2. $H \circ \chi$ is uniformly absolutely continuous with respect to $\mu$.
3. If $a_n \to a$, then $\|\hat{\nu}_{a_n} - \hat{\nu}_a\| \to 0$ uniformly over $\hat{\nu} \in H$, where $\hat{\nu}_b(f) = \hat{\nu}_b(\chi(b)f)$.

Note that (2) applied to $H = \{\hat{\nu}\}$, $\hat{\nu}$ fixed in $L^1(A)$ yields: $\varepsilon > 0 \Rightarrow \exists \delta > 0 \exists \mu(a) < \delta \Rightarrow \nu(a) < \varepsilon$.

There are a number of obvious questions remaining, but in particular: When
and only when is $L^1(A)$ the $L^1$-space of a Boolean measure algebra?

**References**

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA,
MISSOURI 65201