OPERATOR ALGEBRAS AND ALGEBRAIC K-THEORY

LAWRENCE G. BROWN

Communicated by I. M. Singer, June 23, 1975

1. Introduction. We wish to announce several related results which demonstrate a relationship between operator theory and algebraic K-theory. Some of these results concern extensions of C*-algebras (cf. [4], [5]) and complement the results of [4]. Others concern the trace and determinant invariants defined in [7].

2. Extensions of C*-algebras. Let \( H \) be a separable infinite dimensional Hilbert space, \( L(H) \) the algebra of bounded linear operators on \( H \), \( K \) the ideal of compact operators, and \( A = L(H)/K \). In [4] and [5] \( \text{Ext}(X) \) was defined as the set of equivalence classes of C*-algebra extensions, \( 0 \to K \to E \to C(X) \to 0 \), for \( X \) a compact metric space and \( C(X) \) the algebra of continuous complex functions on \( X \). \( \text{Ext}(A) \) was also described as unitary equivalence classes of *-isomorphisms \( \tau: C(X) \to A \). It was shown that \( \text{Ext}(X) \) is a group and that it gives rise to a generalized homology theory which is related to K-theory in roughly the same way as homology is related to cohomology. A Bott periodicity map, \( \text{Per}: \text{Ext}(S^2 X) \to \text{Ext}(X) \), was defined and was proved to be injective for all \( X \) and surjective for smooth \( X \). Also \( \text{Ext}(X) \) was given the structure of a not necessarily Hausdorff topological group, and the closure of the identity was called \( \text{PExt}(X) \).

**Theorem 1.** \( \text{Per} \) is surjective for all \( X \).

**Theorem 2.** There is a natural short exact sequence,

\[
0 \to \text{Ext}_2^1(K^0(X), \mathbb{Z}) \to \text{Ext}(X) \xrightarrow{\gamma} \text{Hom}(K^1(X), \mathbb{Z}) \to 0,
\]

which splits noncanonically.

**Corollary.** \( \text{PExt}(X) \) is the maximum divisible subgroup of \( \text{Ext}(X) \).

**Theorem 3.** If \( \tau_t: C(X) \to A, 0 \leq t \leq 1 \), is a continuous family in the sense that \( \tau_t(f) \) is continuous for each \( f \in C(X) \), then each \( \tau_t \) defines the same element of \( \text{Ext}(X) \).

For a more leisurely account of these results, see [3]. See also [4], [5], [8]. \( \text{Ext} \) satisfies parallel axioms to the Steenrod homology theory [11], whose axiomatic description in [10] plays a key role in the proofs. Algebraic K-theory

---

AMS (MOS) subject classifications (1970). Primary 46L05.

1Research partially supported by a grant from the National Science Foundation.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
(cf. [9]) also plays a key role by yielding a natural definition of an isomorphism $\kappa: \ker \gamma \to \text{Ext}^1_2(K^0(X), \mathbb{Z})$. $\kappa$ is defined by applying the algebraic $K$-theory long exact sequence to $0 \to K \to E \to C(X) \to 0$ and obtaining (in part)

$$0 \to \mathbb{Z} \cong K_0(K) \to K_0(E) \to K_0(C(X)) \cong K^0(X) \to 0.$$ 

We are grateful to J. Milnor, J. Kaminker, and C. Schochet for valuable communications.

In another context [2] we have defined an almost polonais group as the quotient of a polonais (complete, separable, metrizable) group by a normal subgroup which is a continuous homomorphic image of a polonais group. These are not necessarily Hausdorff topological groups with some additional structure, and the abelian ones form an abelian category. Theorem 2 shows that Ext$(X)$ is the direct sum of two almost polonais groups, and we would like to know whether Ext$(X)$ is naturally such an object.

3. The trace and determinant invariants. Let $\mathfrak{A}$ be a $\ast$-subalgebra of $L(H)$ such that $\mathfrak{A}$ contains the trace class, $J$, and is commutative modulo $J$. As in [7], we obtain a symbol map $\phi: \mathfrak{A} \to C(X)$. Here we assume $X \subset \mathbb{R}^n$ and range $\phi = C^\infty(X)$, the algebra of restrictions to $X$ of $C^\infty$ functions on $\mathbb{R}^n$. Let $\overline{X}$ be a closed ball containing $X$. Helton and Howe [7] defined a trace invariant $I: \Omega \to C$, where $\Omega$ is the space of exact $C^\infty$ 2-forms on $\overline{X}$ and $I(df \wedge dg) = \text{tr}(AB - BA)$, where $A$ and $B$ are elements of $\mathfrak{A}$ such that $\phi(A) = f \mid X$ and $\phi(B) = g \mid X$. If $A$ and $B$ are invertible, Helton and Howe also considered $\text{det}(ABA^{-1}B^{-1}) = \delta(\phi(A), \phi(B))$. $\delta$ is a bimultiplicative form on a subgroup of the group of units in $C^\infty(X)$. In [1] we showed, in the special case $X \subset \mathbb{R}^2$, that $\delta$ can be extended to a form $d$ on the whole group of units and that $d$ can be calculated from the trace invariant. As suggested to us by H. Sah, the algebraic properties of $d$ provided an analogy with algebraic $K$-theory. We will now define a new determinant invariant, $d_1: K_2(C^\infty(X)) \to C^\ast$, such that $d$ is the restriction of $d_1$ to the Steinberg symbols.

Consider the short exact sequence, $0 \to J \to \mathfrak{A} \to \mathfrak{A}/J \to 0$, and the corresponding algebraic $K$-theory long exact sequence $\cdots \to K_2(\mathfrak{A}/J) \to K_1(J) \to K_1(\mathfrak{A}) \cdots$. Using the definition of $K_1(J)$ and the most basic properties of the determinant (on the determinant class, $J + I \subset L(H)$), we obtain a map det: $K_1(J) \to C^\ast$. This pulls back to $d$: $K_2(\mathfrak{A}/J) \to C^\ast$. Using analytic techniques (mainly suggested by [7]), we can modify $d'$ to obtain $d_1$.

The restriction of $d_1$ to $K_2'(C^\infty(\overline{X})) \to K_2(\mathfrak{A}/J)$ (which is the same as the kernel of $K_2(C^\infty(X)) \to K^2(X)$), can be calculated from the trace invariant: Roughly one shrinks $\overline{X}$ to a point and differentiates with respect to “time”. In this way we obtain a map $\theta: K_2(C^\infty(\overline{X})) \to \Omega$, and $d_1(C) = \exp(I(\theta(C)))$, for $C \in K_2(C^\infty(\overline{X}))$ and $C$ its image in $K_2(C^\infty(X))$. The above leads to an explicit formula for $\theta$. See [6] for the relation and application of this formula to algebraic $K$-theory. Although $\overline{C}$ is not uniquely determined by
$C$, the restriction of $\theta(\tilde{C})$ to $X$ is unique. If $l$ vanishes at 2-forms which vanish on $X$, then we obtain $l': K_2 \to C$. According to [7], this occurs precisely when $\overline{\gamma}_2 \in \text{Ext}(X)$, and one can then ask whether $l'$ can be extended to $l'': K_2(C^0(X)) \to C$ such that $d_1 = \exp(l'')$. This leads to an element of $\text{Ext}^2(K^0(X), \mathbb{Z})$, which vanishes precisely when $l''$ exists.

**Remarks**

1. Although the construction just completed motivated $\kappa$, we do not know whether the two constructions actually agree.

2. The algebra $\mathfrak{A}$ is what Helton and Howe call a "one dimensional" algebra. It would be nice to extend the above to the "$k$-dimensional" algebras of [7]. There seem to be two difficulties: (a) So far as we know, no existing treatment of $K_n$ for $n > 2$ lends itself to explicit formulas as well as [9]. (b) In the $k$-dimensional case the determinant invariant ought to be defined on $K_{2k}$; but if we do what is natural in the context of [7], we get something on $K_{k+2}$ (for $k > 1$). Thus perhaps something is wrong for $k > 2$.

We hope that these difficulties will eventually be surmounted and that the result will be significant mutual enrichment of operator theory and algebraic $K$-theory.

**References**


2. ———, *Group cohomology of topological groups* (in preparation).


DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907