STABILITY OF EQUIVARIANT SMOOTH MAPS
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This research announcement is a summary of a paper which will appear elsewhere [5], and which continues the program started in [4].

1. We consider a compact Lie group $G$ and smooth compact $G$-manifolds $X$ and $Y$. By $C^\infty_G(X, Y)$, $\text{Diff}_G(X)$, $\text{Diff}_G(Y)$ we denote the $C^\infty$, $G$-equivariant mappings $X \to Y$, respectively, diffeos of $X$ or diffeos of $Y$.

There is a natural group action

$$\text{Diff}_G(X) \times \text{Diff}_G(Y) \times C^\infty(X, Y) \to C^\infty(X, Y),$$

and for each $f \in C^\infty_G(X, Y)$, we define the corresponding orbit-map

$$\text{Diff}_G(X) \times \text{Diff}_G(Y) \to C^\infty_G(X, Y).$$

We consider the $G$-bundles $TX$, $TY$, $f^*TY$ and their “invariant sections” $\Gamma^\infty(TX)^G$, $\Gamma^\infty(TY)^G$, $\Gamma^\infty(f^*TY)^G$. (These are modules over the corresponding rings of $G$-invariant functions.)

As in the usual case [3], [6] we have linear mappings

$$\Gamma^\infty(TX)^G \xrightarrow{\beta_f} \Gamma^\infty(f^*TY)^G$$

$$\Gamma^\infty(TY)^G \xrightarrow{\alpha_f}$$

defined in a natural way.

By definition, $f$ is infinitesimally stable if $\alpha_f + \beta_f$ is surjective.

By definition, $f$ is stable if Image $\Phi_f$ is a neighbourhood of $f \in C^\infty_G(X, Y)$.

With these definitions we have the

**STABILITY THEOREM.** Let $f \in C^\infty_G(X, Y)$ be infinitesimally stable. Then:

(i) Whenever $Z_1$ is the germ of a metrizable or compact topological space, $Z_2$ the germ of a smooth finite dimensional manifold, and $\psi: Z_1 \times Z_2 \to C^\infty_G(X, Y)$ a $C^0,\infty$-germ of a map sending the base points to $f$, there is a germ of a $C^0,\infty$ map $\Psi: Z_1 \times Z_2 \to \text{Diff}_G(X) \times \text{Diff}_G(Y)$ sending the base points to $(\text{id } X) \times (\text{id } Y)$ and such that


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Diff\(_G(X) \times \text{Diff}_G(Y)\)

\[\Psi \xrightarrow{\Phi_f} \psi \xrightarrow{} C^\infty_G(X, Y)\]

is commutative.

(ii) There is a neighbourhood \(f \in N \subset C^\infty_G(X, Y)\) such that every \(f' \in N\) is also infinitesimally stable

(iii) \(f\) is stable. \(\square\)

The proof relies heavily on the work of J. Mather [2], [3] (which is generalized by this theorem) and of G. Schwarz [7].

2. Let \(G \xrightarrow{\psi} \text{Aut}(V)\) be a linear representation of the compact Lie group \(G\). Let \(x \in V\) be the current point of \(V\) and \(R[x]^G \subset R[x]\) be the algebra of \(G\)-invariant polynomials. According to a classical theorem of Hilbert [1], [8] we can always choose a finite system \((\rho_1, \ldots, \rho_k) = \rho \subset R[x]^G\) of algebra generators of \(R[x]^G\). We shall attach to the representation \((G, \psi)\) the number

\[\text{ord}(G, \psi) = \min_{\rho} \left( \max_i \text{deg} \rho_i \right) \in \mathbb{Z}^+\]

Suppose now \(X\) is a (not necessarily compact) \(G\)-manifold (\(G\) compact). By the slice-representation we have a naturally defined function on the space of orbits \(X/G \xrightarrow{\text{ord}} \mathbb{Z}^+\).

One of the technical ingredients occurring in the context of the stability theorem is the following

**Semicontinuity Lemma.** For every orbit \(\{Gx\} \in X/G\) there exists a neighbourhood \(\{Gx\} \in W \subset X/G\), such that, for any \(\{Gx'\} \in W\), one has

\[\text{ord}(Gx') \leq \text{ord}(Gx). \quad \square\]

This might be useful in the study of deformations of group actions suggested by Thom.

**BIBLIOGRAPHY**


