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OPTIMAL LIPSCHITZ AND L^p ESTIMATES FOR THE EQUATION $\bar\partial u = f$ ON STRONGLY PSEUDO-CONVEX DOMAINS[1]

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For definitions and notation in what follows, see Hörmander [5]. Let $\mathcal{D} \subset\subset \mathbf{C}^n$ be strongly pseudo-convex with C^5 boundary. Let

$$\Lambda_\alpha(\mathcal{D}) = \left\{ f : \mathcal{D} \to \mathbf{C} : \|f\|_{L^\infty} + \sup_{z, z+h \in \mathcal{D}} \frac{|f(z) - f(z+h)|}{|h|^\alpha} = \|f\|_{\Lambda_\alpha} < \infty \right\},$$

$$L^p(\mathcal{D}) = \left\{ f : \mathcal{D} \to \mathbf{C} : \int_{\mathcal{D}} |f|^p \, dL < \infty \right\}, \qquad 1 \leq p < \infty,$$

where dL is Lebesgue measure.

We wish to announce Lipschitz and L^p regularity results for Henkin's solution to $\bar\partial u = f$, f a $(0, 1)$ form with $\bar\partial f = 0$, which are essentially best possible, not only for his solution, but for any solution to the equation. More precisely,

THEOREM 1. *There exists a linear operator T taking $\bar\partial$ closed* $(0, 1)$ *forms with coefficients in $C^\infty(\mathcal{D})$ to functions in $C^\infty(\mathcal{D})$ and satisfying*

(a) $\bar\partial Tf = f$,

(b) $\|Tf\|_{L^q} \leq A_p \|f\|_{L^p}$, $1 < p < 2n + 2$, $1/q = 1/p - 1/(2n + 2)$,

(c) $\|Tf\|_{\Lambda_{1/2-(n+1)/p}} \leq A_p \|f\|_{L^p}$, $2n + 2 < p \leq \infty$,

(d) $\|Tf\|_{L^{(2n+2)/(2n+1)-\epsilon}} \leq A_\epsilon \|f\|_{L^1}$, $\epsilon > 0$,

(e) $\int_{\mathcal{D}} \exp(a/\|f\|_{L^{2n+2}} |Tf|^{(2n+2)/(2n+1)}) \, dL \leq C$, *where a, C do not depend on f.*

The constants a, C, A_ϵ, A_p are independent of "small" perturbations of $d\mathcal{D}$.

We give examples to show that

(b′) $\exists \mathcal{D} \subset\subset \mathbf{C}^n$ and $f_p \in C^\infty_{(0,1)}(\mathcal{D})$ such that $\mathcal{D}$ is strongly pseudo-convex, $\|f_p\|_{L^{p-\epsilon}} < \infty$ $\forall \epsilon > 0$, $\bar\partial f_p = 0$, and no u satisfies both $\bar\partial u = f_p$ and $\|u\|_{L^q} < \infty$, $1/q = 1/p - 1/(2n + 2)$, $1 < p < 2n + 2$.

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(c′) $\exists D, f_p$ as above with $\bar\partial f_p = 0$, and no u satisfies both $\bar\partial u = f_p$ and $\|u\|_{\Lambda_{1/2-(n+1)/p+\epsilon}} < \infty$, $\epsilon > 0$, $p > 2n+2$.

That (b), (c) in Theorem 1 are best possible follows immediately. That (d), (e) are best is implicit.

Using some new Lipschitz spaces introduced in [6] by Stein, we are able to prove that not only is Henkin's solution $Tf \in \Lambda_{(1/2)-(n+1)/p}$ when $f \in L^p$, but that Tf restricted to curves all of whose tangents lie in the complex tangential directions is in $\Lambda_{1-(2n+2)/p-\epsilon}$ for all $\epsilon > 0$. Again, examples show that this estimate cannot be improved.

To obtain estimates of the above type, we use the classical technique of obtaining estimates on the integration kernels in Henkin's representation for T (see [4]). The fact that Henkin's integrals are boundary integrals, coupled with the fact that the kernels are neither of the Riesz potential type nor of the non-isotropic type studied by Folland and Stein [1] and others, but rather a product of the two, makes the estimates nontrivial. The proofs require an interpolation theorem and some convexity theorems which we have not found elsewhere in print. Moreover, some new results of Stein about regularity for the Bergman projection operator and estimates involving splitting of the complexified tangent bundle are crucial to the result. Details will appear elsewhere.

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