A SUFFICIENT CONDITION
FOR \( k \)-PATH HAMILTONIAN DIGRAPHS
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A directed graph (or digraph) \( D \) is: (1) \textit{traceable} if \( D \) has a hamiltonian path; (2) \textit{hamiltonian} if \( D \) has a hamiltonian cycle; (3) \textit{strongly hamiltonian} if \( D \) has arcs and each arc lies on a hamiltonian cycle; (4) \textit{hamiltonian-connected} if \( D \) has a hamiltonian \( u \)-\( v \) path for every pair of distinct vertices \( u \) and \( v \); (5) \textit{\( k \)-path traceable} if every path of length not exceeding \( k \) is contained in a hamiltonian path; and (6) \textit{\( k \)-path hamiltonian} if every path of length not exceeding \( k \) is contained in a hamiltonian cycle.

The indegree and the outdegree of a vertex \( v \) are denoted by \( \text{id}(v) \) and \( \text{od}(v) \) respectively. A digraph \( D \) of order \( p \) is of Ore-type (\( k \)) if \( \text{od}(u) + \text{id}(u) > p + k \) whenever \( u \) and \( v \) are distinct vertices for which \( uv \) is not an arc of \( D \).

In this research announcement we outline a proof of the following result, a complete proof of which will appear elsewhere, and present some consequences of it.

**THEOREM.** If a nontrivial digraph \( D \) is of Ore-type (\( k \)), \( k \geq 0 \), then \( D \) is \( k \)-path hamiltonian.

**PROOF.** Let \( D \) have order \( p \geq 2 \). First, observe that \( D \) is strong. Since the result holds if \( D \) is the complete symmetric digraph \( K_p \), we assume that \( D \neq K_p \). This in turn implies that \( p \geq k + 4 \). Also, it can be shown that every path of length not exceeding \( k \) is contained in a path of length \( (k + 1) \) and this longer path is contained in a cycle.

Suppose \( D \) has a path \( P: v_1, v_2, \ldots, v_{k+1} \) of length \( k \) which is contained in no hamiltonian cycle. Let \( C: v_1, v_2, \ldots, v_n, v_1 \) be any longest cycle containing \( P \). Then, \( V = V(D) - V(C) \neq \emptyset \), where \( V(D) \) and \( V(C) \) denote the vertex sets of \( D \) and \( C \) respectively.

Now, assume that \( V \) has distinct vertices \( u \) and \( v \) for which \( uv \notin E(D) \) and the subdigraph \( \langle V \rangle \) induced by \( V \) has no \( u \)-\( v \) path. Then, \( uu \notin E(D) \) implies that

\[
p + k - \text{od}(u) + \text{id}(u) \leq p - n - 2 + \text{od}(v, C) + \text{id}(u, C)
\]

where \( \text{od}(u, C) \) and \( \text{id}(u, C) \) denote the number of vertices in \( C \) which are
dominated by $v$ and dominate $u$, respectively. Then (1) implies that $n + k + 2 \leq \text{od}(v, C) + \text{id}(u, C)$ and this implies that $\langle V \rangle$ has no $u-v$ path. For suppose that $\langle V \rangle$ has such a path. Since $C$ is a longest cycle containing $P$, the digraph $D$ cannot contain both of the arcs $v_i u$ and $v_{i+1} u$ for $k + 1 \leq i \leq n$. But this implies that $\text{id}(u, C) + \text{od}(v, C) \leq n + k$ and this is a contradiction. Using the fact that $uv \notin E(D)$, we obtain

$$p + k \leq \text{od}(u) + \text{id}(v) \leq p - n - 2 + \text{od}(u, C) + \text{id}(v, C)$$

which also implies that $n + k + 2 \leq \text{od}(u, C) + \text{id}(v, C)$. Together with the preceding result, this implies that either

$$n + k + 2 \leq \text{od}(u, C) + \text{id}(u, C) \quad \text{or} \quad n + k + 2 \leq \text{od}(v, C) + \text{id}(u, C).$$

In either case, it follows that $D$ has a longer cycle containing $P$ which is impossible. Thus, for distinct vertices $u$ and $v$ of $\langle V \rangle$, either $uv \in E(\langle V \rangle)$ or $\langle V \rangle$ has a $v-u$ path. If $\langle V \rangle$ has a $v-u$ path, then $\text{od}(u, C) + \text{id}(v, C) \leq n + k$. Thus,

$$\text{od}(u, \langle V \rangle) + \text{id}(v, \langle V \rangle) \geq p - n = |V|$$

whenever $u \neq v$ and $uv \notin E(\langle V \rangle)$. Hence, $\langle V \rangle$ is strongly connected.

Let $W$ be the subpath $v_{k+1}, v_{k+2}, \ldots, v_n, v_{n+1} = v_1$ of $C$. Since $n \geq k + 2$, the path $W$ has order at least 3; in fact $W$ has at least 3 vertices which are dominated by vertices of $V$ and at least 3 vertices which dominate vertices of $V$. It now suffices to consider the following two cases: (i) the path $W$ has a non-trivial subpath $W'$ whose initial vertex dominates a vertex of $V$ and whose terminal vertex is dominated by a vertex of $V$; and (ii) the path $W$ has no such subpath. Since consideration of either case leads to contradiction, our assumption that $V \neq \emptyset$ must be false. Hence, $C$ is a hamiltonian cycle and the theorem follows.

Let $m, n \geq 1$ and $k \geq 0$. The symmetric join $K_{k+2} + (K_m \cup K_n)$ of $K_{k+2}$ and the disjoint union of $K_m$ and $K_n$ is an Ore-type $(k)$ digraph which is not $(k + 1)$-path hamiltonian. Hence, the result is “best possible.”

The preceding result generalizes several results from graph theory and digraph theory, which we present below.

**Corollary.** If the digraph $D$ is of Ore-type $(k)$, $k \geq -1$, then $D$ is $(k + 1)$-path traceable.

**Corollary (Woodall [5]).** If a nontrivial digraph is of Ore-type $(0)$, then it is hamiltonian.

**Corollary.** If a nontrivial digraph is of Ore-type $(1)$, then it is both strongly hamiltonian and hamiltonian-connected.

A (undirected) graph of order $p$ is of Ore-type $(k)$ if the sum of the degrees
of nonadjacent vertices is at least \((p + k)\). By considering symmetric digraphs, we obtain the following results.

**Corollary (Ore [3]).** *If a graph with order at least 3 is of Ore-type (0), then it is hamiltonian.*

**Corollary (Ore [4]).** *If a graph is of Ore-type (1), then it is hamiltonian-connected.*

**Corollary (Kronk [2]).** *If a graph of order \(p \geq 3\) is of Ore-type \((k)\), \(k \geq 0\), then it is \(k\)-path hamiltonian.*

**Corollary (Kapoor and Theckedath [1]).** *If a graph is of Ore-type \((k)\), \(k \geq -1\), then it is \((k+1)\)-path traceable.*

**REFERENCES**


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