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ON THE EXISTENCE AND UNIQUENESS OF STREBEL DIFFERENTIALS

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Let X be a compact Riemann surface of genus $g > 1$, and $q \in H^0(X, \Omega^{\otimes 2})$ be a holomorphic quadratic form on X. A tangent vector $\xi \in T_x X$ is called horizontal if $\langle q, \xi \otimes \xi \rangle > 0$. The horizontal vectors define a foliation of X singular at the zeroes of q. The form q is called a *Strebel form* if the leaves of this foliation are compact.

If q is a Strebel form, the leaves of the foliation through a zero of q form a graph Γ_q, and $X - \Gamma_q$ is a union of metric straight cylinders (for the metric $|q|^{1/2}$). The central circles in each cylinder form a set of disjoint, nonpairwise homotopic and homotopically nontrivial simple closed curves on X, called the system of curves associated to q.

Let M be an oriented differentiable compact surface of genus g, and $\mathcal{C}$ a system of n simple closed curves on M, disjoint, not pairwise homotopic and homotopically nontrivial. In the vector bundle Q of pairs (θ, q), with θ in the Teichmüller space Θ_M (see [2] for notation) and q a quadratic form on the Riemann surface above θ, consider the space $E_{\mathcal{C}} \subset Q$ of Strebel forms whose associated system of curves is homotopic to $\mathcal{C}$. Denote $\pi: E_{\mathcal{C}} \longrightarrow \Theta_M \times \mathbf{R}^n_+$ the map whose first factor is the canonical projection, and whose second factor gives the heights of the cylinders. Our main result is the following

THEOREM. *The map* $\pi: E_{\mathcal{C}} \longrightarrow \Theta_M \times \mathbf{R}^n_+$ *is a homeomorphism.*

A similar result was proved by Strebel [4], with a very different proof. Further information can be found in [1], [4] and [5]. We wish to thank A. Douady for his most valuable help.

It is easy to check that a point q in $E_{\mathcal{C}}$ is completely determined by the homotopy class of its critical graph Γ_q in M, the lengths of the segments of Γ_q, the heights of the cylinders and parameters measuring "the twisting around the central circles" of each cylinder. This allows an elementary, geometric and useful [3] description of Riemann surfaces.

The proof of the theorem proceeds in three steps: proving that π is proper that π is a local homeomorphism, and that $E_{\mathcal{C}}$ is connected. The result then

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follows from the fact that Θ_M is simply connected. We shall give below an outline of the proof of each step; details will appear elsewhere.

PROPOSITION 1. *The map π is proper.*

Consider the union $E = \bigcup E_{C'}$, where C' is a subset of C. Since integral curves of vector fields depend continuously on the vector field, E is closed in Q. The map π can be extended continuously to E by assigning height zero to degenerate cylinders. The proposition then follows from the fact that the unit sphere bundle in Q is proper over Θ_M, and the following lemma:

LEMMA. *If Γ is a homotopy class of closed curves on M, the function $q \longmapsto \inf_{\gamma \in \Gamma} \int_\gamma |q|^{1/2}$ is continuous on Q.*

The space E_k. Denote by P_k the space of polynomial quadratic forms on $\mathbf{C}$ of the form $(z^k + p(z))\,dz^2$, with p a polynomial of degree at most $k - 2$. It is easily checked that P_k is a versal deformation of $z^k\,dz^2$ near $z = 0$. Let $E_k \subset P_k$ be the set of quadratic forms q with connected critical graph Γ_q. For any $x \neq 0$ in $\mathbf{C}$ the function $f(q) = \lim \int_x^{\Gamma_q} \sqrt{q}$ is well defined near $p =\backslash 0$ in E_k. Embed E_k in $P_k \times \mathbf{R}$ by $q \longmapsto (q, f(q))$.

PROPOSITION 2. *The image of E_k in $P_k \times \mathbf{R}$ is a differentiable submanifold of $P_k \times \mathbf{R}$ near 0. The tangent space T_0E_k is the set of pairs (p, s) where; if k is even, p is a polynomial whose coefficients of degree $< k/2$ vanish, and s is arbitrary; and if k is odd, the coefficients of degree $< (k - 1)/2$ vanish, and $s = \frac{1}{2} \operatorname{Im} \int_0^x p/z^{k/2}\,dz$.*

The main step in the proof is to show that if $p(z)$ is tangent to E_k at q, and q has simple zeroes, then p must have nonzero coefficients above the middle degree. On the Riemann surface of $\sqrt{q}$, forms $p/\sqrt{q}$ with $\deg p \leqslant [(k - 3)/2]$ form a basis for the holomorphic differentials, and the integrals of such forms over curves covering the bounded segments of Γ_q cannot all be real. The result then follows from induction on k and from the homogeneity of E_k.

The map π is a local homeomorphism. Let $q_0 \in E_C$ vanish at points $x_1, \ldots, x_m$ to orders $k_1, \ldots, k_m$. Then a neighborhood U of q in Q parametrizes deformations of the zeroes of q_0, and we get a map $U \longrightarrow \prod P_{k_i}$ classifying these deformations. The fibre product V of U and $\prod E_{k_i}$ over $\prod P_{k_i}$ is a differentiable manifold (but not a submanifold of Q), and parametrizes the deformations of q "locally Strebel" near the zeroes of q. Because of the last coordinate in E_k, the functions $q \longmapsto \operatorname{Im} \int_{\Gamma_j}^{\Gamma_i} \sqrt{q}$ are differentiable on V, where Γ_i is the critical graph of q near x_i, and the integral is over a path near a segment of Γ_{q_0}. The equations $\operatorname{Im} \int_{\Gamma_j}^{\Gamma_i} \sqrt{q} = 0$ over all such segments define E as a submanifold of $Q \times \mathbf{R}_+^n$, the last coordinate being heights.

PROPOSITION 3. *The map π is differentiable, and its derivative is an isomorphism.*

The proof depends on a decomposition of $H^1(X, T_X)$ into those deformations leaving the zeroes of q unchanged, and deformations with support in small neighborhoods of the x_i's.

PROPOSITION 4. *The space E_C is connected.*

The proof is by induction on the number of curves in C. It uses Lemma 1 and the following result [4]:

LEMMA 2. *Let q be a Strebel differential on X, determining annuli $A_1, \ldots, A_n$ of moduli $M_1, \ldots, M_n$ and circumferences $a_1, \ldots, a_n$ (with respect to $|q|^{1/2}$). Let $B_1, \ldots, B_n$ be disjoint annuli on X, of moduli $N_1, \ldots, N_n$, and homotopic to $A_1, \ldots, A_n$ respectively. Then $\Sigma a_i^2 M_i \geqslant \Sigma a_i^2 N_i$, and equality occurs only if $A_i = B_i$, $i = 1, \ldots, n$.*

In case C consists of a maximal system of $3g - 3$ curves, we are able to characterize the graph in terms of the heights and circumstances of the cylinders, and prove Proposition 4 directly.

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