ON THE EXISTENCE AND UNIQUENESS OF STREBEL DIFFERENTIALS

BY JOHN H. HUBBARD\textsuperscript{1,2} AND HOWARD MASUR\textsuperscript{1}

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Let $X$ be a compact Riemann surface of genus $g > 1$, and $q \in H^0(X, \Omega^2)$ be a holomorphic quadratic form on $X$. A tangent vector $\xi \in T_x X$ is called horizontal if $\langle q, \xi \otimes \xi \rangle > 0$. The horizontal vectors define a foliation of $X$ singular at the zeroes of $q$. The form $q$ is called a Strebel form if the leaves of this foliation are compact.

If $q$ is a Strebel form, the leaves of the foliation through a zero of $q$ form a graph $\Gamma_q$, and $X - \Gamma_q$ is a union of metric straight cylinders (for the metric $|q|^{1/2}$). The central circles in each cylinder form a set of disjoint, nonpairwise homotopic and homotopically nontrivial simple closed curves on $X$, called the system of curves associated to $q$.

Let $M$ be an oriented differentiable compact surface of genus $g$, and $C$ a system of $n$ simple closed curves on $M$, disjoint, not pairwise homotopic and homotopically nontrivial. In the vector bundle $Q$ of pairs $(\theta, q)$, with $\theta$ in the Teichmüller space $\Theta_M$ (see [2] for notation) and $q$ a quadratic form on the Riemann surface above $\theta$, consider the space $E_C \subset Q$ of Strebel forms whose associated system of curves is homotopic to $C$. Denote $\pi: E_C \rightarrow \Theta_M \times \mathbb{R}^n_+$ the map whose first factor is the canonical projection, and whose second factor gives the heights of the cylinders. Our main result is the following

\textbf{Theorem.} The map $\pi: E_C \rightarrow \Theta_M \times \mathbb{R}^n_+$ is a homeomorphism.

A similar result was proved by Strebel [4], with a very different proof. Further information can be found in [1], [4] and [5]. We wish to thank A. Douady for his most valuable help.

It is easy to check that a point $q$ in $E_C$ is completely determined by the homotopy class of its critical graph $\Gamma_q$ in $M$, the lengths of the segments of $\Gamma_q$, the heights of the cylinders and parameters measuring “the twisting around the central circles” of each cylinder. This allows an elementary, geometric and useful [3] description of Riemann surfaces.

The proof of the theorem proceeds in three steps: proving that $\pi$ is proper that $\pi$ is a local homeomorphism, and that $E_C$ is connected. The result then

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follows from the fact that $\Theta_M$ is simply connected. We shall give below an outline of the proof of each step; details will appear elsewhere.

**Proposition 1.** The map $\pi$ is proper.

Consider the union $E = \bigcup E_C'$, where $C'$ is a subset of $C$. Since integral curves of vector fields depend continuously on the vector field, $E$ is closed in $Q$. The map $\pi$ can be extended continuously to $E$ by assigning height zero to degenerate cylinders. The proposition then follows from the fact that the unit sphere bundle in $Q$ is proper over $\Theta_M$, and the following lemma:

**Lemma.** If $\Gamma$ is a homotopy class of closed curves on $M$, the function $q \mapsto \inf_{\gamma \in \Gamma} \int_{\gamma} |q|^{1/2}$ is continuous on $Q$.

The space $E_k$. Denote by $P_k$ the space of polynomial quadratic forms on $C$ of the form $(z^k + p(z)) \, dz^2$, with $p$ a polynomial of degree at most $k - 2$. It is easily checked that $P_k$ is a versal deformation of $z^k \, dz^2$ near $z = 0$. Let $E_k \subset P_k$ be the set of quadratic forms $q$ with connected critical graph $\Gamma_q$. For any $x \neq 0$ in $C$ the function $f(q) = \lim_{q} \int_{x}^{q} \sqrt{q}$ is well defined near $p = 0$ in $E_k$. Embed $E_k$ in $P_k \times \mathbb{R}$ by $q \mapsto (q, f(q))$.

**Proposition 2.** The image of $E_k$ in $P_k \times \mathbb{R}$ is a differentiable submanifold of $P_k \times \mathbb{R}$ near $0$. The tangent space $T_0 E_k$ is the set of pairs $(p, s)$ where; if $k$ is even, $p$ is a polynomial whose coefficients of degree $< k/2$ vanish, and $s$ is arbitrary; and if $k$ is odd, the coefficients of degree $< (k - 1)/2$ vanish, and $s = \frac{1}{2} \Im \int_{x}^{q} p(z) \, dz$.

The main step in the proof is to show that if $p(z)$ is tangent to $E_k$ at $q$, and $q$ has simple zeroes, then $p$ must have nonzero coefficients above the middle degree. On the Riemann surface of $\sqrt{q}$, forms $p/\sqrt{q}$ with $\deg p \leq [(k - 3)/2]$ form a basis for the holomorphic differentials, and the integrals of such forms over curves covering the bounded segments of $\Gamma_q$ cannot all be real. The result then follows from induction on $k$ and from the homogeneity of $E_k$.

The map $\pi$ is a local homeomorphism. Let $q_0 \in E_C$ vanish at points $x_1, \ldots, x_m$ to orders $k_1, \ldots, k_m$. Then a neighborhood $U$ of $q_0$ in $Q$ parametrizes deformations of the zeroes of $q_0$, and we get a map $U \to \Pi P_{k_i}$ classifying these deformations. The fibre product $V$ of $U$ and $\Pi P_{k_i}$ over $\Pi P_{k_i}$ is a differentiable manifold (but not a submanifold of $Q$), and parametrizes the deformations of $q$ "locally Strebel" near the zeroes of $q$. Because of the last coordinate in $E_k$, the functions $q \mapsto \Im \int_{\Gamma_i} \sqrt{q}$ are differentiable on $V$, where $\Gamma_i$ is the critical graph of $q$ near $x_i$, and the integral is over a path near a segment of $\Gamma_q$. The equations $\Im \int_{\Gamma_i} \sqrt{q} = 0$ over all such segments define $E$ as a submanifold of $Q \times \mathbb{R}^n$, the last coordinate being heights.

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PROPOSITION 3. The map $\pi$ is differentiable, and its derivative is an isomorphism.

The proof depends on a decomposition of $H^1(X, T_X)$ into those deformations leaving the zeroes of $q$ unchanged, and deformations with support in small neighborhoods of the $x_i$'s.

PROPOSITION 4. The space $E_C$ is connected.

The proof is by induction on the number of curves in $C$. It uses Lemma 1 and the following result [4]:

**Lemma 2.** Let $q$ be a Strebel differential on $X$, determining annuli $A_1, \ldots, A_n$ of moduli $M_1, \ldots, M_n$ and circumferences $a_1, \ldots, a_n$ (with respect to $|q|^{1/2}$). Let $B_1, \ldots, B_n$ be disjoint annuli on $X$, of moduli $N_1, \ldots, N_n$, and homotopic to $A_1, \ldots, A_n$ respectively. Then $\Sigma a_i^2 M_i \geq \Sigma a_i^2 N_i$, and equality occurs only if $A_i = B_i, i = 1, \ldots, n$.

In case $C$ consists of a maximal system of $3g - 3$ curves, we are able to characterize the graph in terms of the heights and circumstances of the cylinders, and prove Proposition 4 directly.

BIBLIOGRAPHY


DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MASSACHUSETTS 02138