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EIGENVALUES ASSOCIATED WITH A CLOSED GEODESIC[1]

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1. **Background.** Intuitive arguments drawn from quantum mechanics and optics suggest that there should be some relation between the closed geodesics (periodic "particles") on a compact Riemannian manifold X and the eigenvalues (periodic "waves") of the Laplace-Beltrami operator Δ_X. Indeed, in 1959, Huber [8] proved, for X a surface of constant negative curvature, that the set of lengths of closed geodesics on X and the spectrum of Δ_X determine one another. The relation given by Huber between these two sequences of numbers is sufficiently complicated to make it extremely difficult to find one sequence explicitly, given the other.

Recently, Colin de Verdière [2], then Chazarain [1] and Duistermaat and Guillemin [3] have shown that, for most Riemannian metrics on any differentiable manifold, the spectrum of the Laplacian determines the lengths of the closed geodesics and their Morse indices modulo 4. Here, the lengths of the closed geodesics appear as the singular points of the distribution

$$\hat{\sigma}(t) = \mathrm{Trace}(e^{it\sqrt{\Delta_X}}) = \sum_{\lambda_j \in \mathrm{Spec}\, \Delta_X} e^{i\sqrt{\lambda_j}\, t}$$

on the real line. This result, although very striking, leaves open some important questions. To apply it in any particular case, one would need to know a lot about Spec Δ_X to get any information about the closed geodesics; even then, a formidable calculation would be involved in all but the simplest cases. In fact, when one is "handed" a Riemannian manifold, it is more likely that one knows something about the closed geodesics than about the spectrum.

In quantum mechanics, the eigenvalues are energy levels, and any available information about them is of interest. In a series of four papers culminating in [6], Gutzwiller used the method of Feynman integrals to find a contribution to the spectral density distribution

$$\sigma(\lambda) = \sum_{\lambda_j \in \mathrm{Spec}\, \Delta_X} \delta(\lambda - \lambda_j)$$

corresponding to a single closed geodesic γ. For stable γ, this contribution is a series of δ-functions at locations which are presumably approximate eigenvalues;

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for γ unstable, the contribution is a series of smooth peaks. Gutzwiller's arguments are quite physical and make no pretense of rigor, but his results have been very suggestive.

Recently, Voros [12] has clarified Gutzwiller's work in the case of stable closed geodesics and has provided simpler, though still heuristic, arguments based on the Keller-Maslov correction of the Bohr-Sommerfeld quantization condition. Essentially, Voros replaces the given classical system by its completely integrable linearization at the closed geodesic (in the cotangent bundle T^*X) and sees which of the invariant tori satisfy the quantization condition.

The purpose of the present note is to announce a proof of Voros' result, based upon an extension of Hörmander's calculus of Fourier integral operators [5], [11].

2. **Statement of Theorem.** Before stating our theorem, we introduce some terminology and notation. Let γ be a closed geodesic on the $k+1$ dimensional manifold X. Denote by P_γ the linearized Poincaré map of γ; it is a symplectic automorphism of the tangent space V_γ of a $2k$-dimensional manifold transverse to γ in the unit sphere bundle in T^*X. γ is called *nondegenerate elliptic* if the eigenvalues of P_γ are all distinct and have modulus one. The map P_γ then splits into a direct sum of two-dimensional rotations through angles $\theta_1, \ldots, \theta_k$ between 0 and 2π. The *Morse index* of γ is the index of γ as a critical point of the length functional on the loop space of γ. Finally, our proof requires the assumption that T^*X admits a metaplectic structure [5]. This is a very weak requirement, satisfied for example if X is orientable, and probably not really necessary for the result to be true.

THEOREM. *If γ is a nondegenerate elliptic closed geodesic on X with length L, rotation angles $\theta_1, \ldots, \theta_k$, and Morse index μ, then for every k-tuple $(n_1, \ldots, n_k)$ of nonnegative integers there exists a sequence $\lambda_1, \lambda_2, \ldots$ of eigenvalues of Δ_X satisfying*

$$\sqrt{\lambda_n} = L^{-1}(n_1\theta_1 + \ldots + n_k\theta_k + 2\pi n + \mu) + O(n^{-1/2}).$$

REMARK. If one drops the term $O(n^{-1/2})$ in (*) and computes the resulting contribution from all these eigenvalues to $\hat{\sigma}(t)$, one obtains a pole at $t = L$ whose residue is exactly that calculated in [3]. It remains to be seen whether the singularities of $\hat{\sigma}(t)$ corresponding to unstable closed geodesics can also be traced to the spectrum itself.

3. **Comments on the proof.** The idea for the proof comes from a technique used by one of us [13] to demonstrate a theorem of Moslov [10] relating a sequence of eigenvalues of Δ_X to a Lagrangian submanifold L contained in the unit sphere bundle of T^*X. (It is this theorem which Voros applies to the invariant tori.) The technique involves constructing out of L a *conic* Lagrangian submanifold $\Lambda \subset T^*X \times T^*S^1$ and considering an associated Fourier integral

operator as an approximate intertwining operator between Δ_{S^1} and Δ_X. It was clear that an extension of the Fourier integral operator calculus from Lagrangian to isotropic submanifolds would permit an extension of Maslov's result to the case where L is isotropic and invariant under the geodesic flow. One such extension is provided by the complex Fourier integral operators of Melin and Sjöstrand [11], which leads to the part of our Theorem with $(n_1, \ldots, n_k) = 0$. Another is given by the calculus developed in Guillemin [5]. The symbols of the intertwining operators in this calculus turn out to be the symplectic spinors [9] on V_γ which are invariant under P_γ. This accounts for the presence of the rotation angles in (*) and the restriction of the Theorem to metalinear manifolds.

Further details of our proof may be found in the last section of [5].

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