DIRECT SUM PROPERTIES OF QUASI-INJECTIVE MODULES
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Abstract. A functorial method is described by which certain problems can be transferred from quasi-injective modules to nonsingular injective modules. Applications include the uniqueness of nth roots: If \( A \) and \( B \) are quasi-injective modules such that \( A^n \cong B^n \), then \( A \cong B \).

All rings in this paper are associative with unit, all modules are unital right modules, and endomorphism rings act on the left. The letter \( R \) denotes a ring. We use \( J(-) \) to denote the Jacobson radical.

Recall that a module \( A \) is quasi-injective provided any homomorphism of a submodule of \( A \) into \( A \) extends to an endomorphism of \( A \). For example, all injective modules and all semisimple (completely reducible) modules are quasi-injective.

**Theorem 1.** Let \( A \) be a quasi-injective right \( R \)-module, and set \( Q = \text{End}_R(A) \). Then \( Q/J(Q) \) is a regular, right self-injective ring, and idempotents can be lifted modulo \( J(Q) \).

**Proof.** Regularity and idempotent-lifting were proved by Faith and Utumi [2, Theorems 3.1, 4.1]. Self-injectivity was proved by Osofsky [6, Theorem 12] and Renault [7, Corollaire 3.5].

**Proposition 2.** Let \( A \) be a quasi-injective right \( R \)-module, and set \( Q = \text{End}_R(A) \). Let \( \mathcal{U} \) denote the category of all direct summands of finite direct sums of copies of \( A \), and let \( \mathcal{P} \) denote the category of all finitely generated projective right \( (Q/J(Q)) \)-modules. Then there exists an additive (covariant) functor \( F: \mathcal{U} \to \mathcal{P} \) with the following properties.

(a) For all \( B, C \in \mathcal{U} \), the induced map \( \text{Hom}_R(B, C) \to \text{Hom}_P(F(B), F(C)) \) is surjective.

(b) Given any \( P \in \mathcal{P} \), there exists \( B \in \mathcal{U} \) such that \( F(B) \cong P \).

(c) A map \( f \in \mathcal{U} \) is an isomorphism if and only if \( F(f) \) is an isomorphism in \( \mathcal{P} \).

**Proof.** If \( P_0 \) denotes the category of all finitely generated projective right \( Q \)-modules, then \( \text{Hom}_R(A, -) \) defines a category equivalence \( G: \mathcal{U} \to P_0 \). Second, \( (-) \otimes_Q Q/J(Q) \) gives us an additive functor \( H: P_0 \to \mathcal{P} \), and we set \( F = HG \).
Properties (a) and (c) hold without any hypotheses on \( A \), while (b) follows from the regularity of \( Q/J(Q) \) and the fact that idempotents lift modulo \( J(Q) \).

Over a regular, right self-injective ring, all finitely generated projective right modules are injective and nonsingular. Thus the functor \( F \) in Proposition 2 enables us to transfer problems from the quasi-injective module \( A \) to the nonsingular injective module \( F(A) \).

**Theorem 3.** Let \( A, B \) be quasi-injective right \( R \)-modules, and let \( n \) be a positive integer.

(a) If \( A^n \) is isomorphic to a direct summand of \( B^n \), then \( A \) is isomorphic to a direct summand of \( B \).

(b) If \( A^n \cong B^n \), then \( A \cong B \).

**Proof.** Setting \( Q = \text{End}_R(B) \), we use Proposition 2 to transfer the problem to nonsingular injective right \((Q/J(Q))-\)modules, where the required properties follow from [5, Proposition 9.1].

**Definition.** A module \( A \) is directly finite provided \( A \) is not isomorphic to any proper direct summand of itself.

**Theorem 4** [1, Proposition 5]. Let \( A \) be a directly finite quasi-injective right \( R \)-module. If \( B \) and \( C \) are any right \( R \)-modules such that \( A \otimes B \cong A \otimes C \), then \( B \cong C \).

**Proof.** If \( P \) is any directly finite nonsingular injective module, then [8, Corollary 8] (or [5, Theorem 3.8]) shows that isomorphic direct summands of \( P \) have isomorphic complements. Using Proposition 2, the module \( A \) has the same property. In addition, [3, Theorem 3] shows that \( A \) has the exchange property, hence cancellation follows from [4, Theorem 2].

**Corollary 5.** If \( A_1, \ldots, A_n \) are directly finite quasi-injective right \( R \)-modules, then \( A_1 \oplus \cdots \oplus A_n \) is directly finite (but not necessarily quasi-injective).

**Proof.** Obviously cancellation carries over from the \( A_i \) to their direct sum. On the other hand, \( \mathbb{Z}/2\mathbb{Z} \) and \( Q \) are directly finite quasi-injective \( \mathbb{Z} \)-modules whose direct sum is not quasi-injective.

**Theorem 6.** If \( A \) is a quasi-injective right \( R \)-module, then there exists a decomposition \( A = B \oplus C \) such that \( B \) is directly finite and \( C \cong C^2 \).

**Proof.** The corresponding decomposition for nonsingular injective modules is given by [5, Proposition 8.4 and Theorem 7.2].

**Corollary 7.** Let \( A \) be a quasi-injective right \( R \)-module. Then \( A \) is directly finite if and only if \( A \) has no nonzero direct summands \( C \) for which \( C \cong C^2 \).
REFERENCES


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