THE FAILURE OF SPECTRAL ANALYSIS IN $L^p$ FOR $0 < p < 1$

BY KAREL de LEEUW

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1. Introduction. For $0 < p < 1$, $L^p$ is the space of measurable $f$ on the circle group $T$ with

$$
\|f\|_p = \left[ (2\pi)^{-1} \int_{-\pi}^{+\pi} |f(x)|^p \, dx \right]^{1/p} < \infty.
$$

If $0 < p < 1$, $L^p$ is not a Banach space, but is a metric space with distance defined by $d(f, g) = \|f - g\|_p$.

A linear subspace of $L^p$ will be called a $T$-subspace if and only if it is closed and translation invariant. If $F$ is a function or a collection of functions in $L^p$, then $L^p(F)$ will denote the smallest $T$-subspace of $L^p$ containing $F$, the $T$-subspace of $L^p$ generated by $F$. If $F = \{e^{i\lambda n} : n \in \Delta\}$, is a collection of exponential functions, $L^p(F)$ will also be denoted by $L^p(\Delta)$.

For $p > 1$, the classification of the $T$-subspaces of $L^p$ is straightforward (see [3, Chapter 11]). The map

$$(1.1) \quad \Delta \rightarrow L^p(\Delta)$$

gives a 1-1 correspondence between the collection of all subsets of integers and all $T$-subspaces of $L^p$.

The purpose of this note is to point out that the case $0 < p < 1$ is much more intricate, to be specific, the map (1.1) is neither 1-1 nor onto. We shall outline proofs of results which imply the following.

**Theorem 1.** Let $0 < p < 1$. Then

(i) $L^p$ has nontrivial $T$-subspaces containing no exponentials;

(ii) There are distinct sets $\Delta$ and $\Gamma$ of integers with $L^p(\Delta) = L^p(\Gamma)$.

Details will be published elsewhere. In what follows, “Proof” should of course be interpreted to mean “Outline of Proof”.

2. Spectral analysis in $H^p$ for $0 < p < 1$; Cauchy integrals. Here we restrict to the $T$-subspace $L^p(\{e^{i\lambda n} : n \geq 0\})$, which is denoted by $H^p$. (For the basic properties of $H^p$ which we use in what follows, see [2, Chapter 7], [4, Chapter 3] or [1].) $H^p$ can also be characterized as follows: Let $D$ be the unit disk $\{z : |z| < 1\}$. We define $H^p(D)$ to consist of all functions $F$ which are analytic in $D$ with $\|F\|_p = \sup\{\|F_r\|_p : 0 < r < 1\} < \infty$, where each $F_r$ is de-
fined on $T$ by $F_r(e^{i\theta}) = F(re^{i\theta})$. The functions in $H^p(D)$ have boundary values a.e. on $T$ and the mapping $F \mapsto \tilde{F}$ defined by $\tilde{F}(e^{i\theta}) = \lim_{r \to 1} F(re^{i\theta})$, a.e. $e^{i\theta} \in T$, is an isometry of $H^p(D)$ onto $H^p$.

We shall denote by $L^p_0$ the $T$-subspace $H^p \cap H^p = \{ f : f$ and $\tilde{f}$ are in $H^p \}$. Propositions 2.2 and 2.3 below show that $L^p_0$ is quite large if $p < 1$ even though it consists only of constant functions if $p \geq 1$. We first indicate how $L^p_0$ is our "universal counterexample" to spectral analysis.

**Proposition 2.1.** $L^p_0$ contains no nonconstant exponential functions.

**Proof.** We may assume $p < 1$. If $H^p \cap H^p$ contained a nonconstant exponential function, it would contain some $e^{in\cdot}$ for $n < 0$. By Theorem 7.35 of [5], $H^p \cap L^1 = H^1$. But $e^{in\cdot} \notin H^1$.

The above proof of course yields a great ideal more than asserted by Proposition 2.1, namely, that $L^p_0 \cap L^1$ consists only of constant functions.

If $\mu$ is a finite Borel measure on $T$, we define $F_\mu$ by

$$ (2.1) \quad F_\mu(z) = \int \frac{w}{w - z} \, d\mu(w), \quad |z| \neq 1. $$

The restriction of $F_\mu$ to the unit disk $D$ will be denoted by $C_\mu$. By Theorem 3.5 of [1], $C_\mu$ is in each $H^p(D)$ for $p < 1$ and thus its boundary function $\tilde{C}_\mu$ is in each $H^p$ for $p < 1$. We will call $\tilde{C}_\mu$ the Cauchy transform of $\mu$.

**Proposition 2.2.** Let $0 < p < 1$. Then $L^p_0$ contains the Cauchy transforms of all singular measures on $T$.

**Proof.** Let $\mu$ be a singular measure on $T$. Define $F_\mu$ by (2.1) so $C_\mu = F_\mu$ on $D$. Then, for $z = re^{i\theta}, |z| < 1, F(z) - F(1/\bar{z}) = \sum_{n=1}^\infty \hat{\mu}(n) r^n e^{in\theta}$, which is the $r$th Abel mean of the Fourier series of $\mu$. Since $\mu$ is singular, Theorem 1.2 of [1] shows that the series converges a.e. in $T$ to $0$. Thus, the function $G$ defined in $D$ by $G(z) = F(1/\bar{z})$ has boundary values conjugate to $\tilde{C}_\mu$ a.e. on $T$. It remains to show that $G \in H^p(D)$ for each $p < 1$. Since $G(z) = -\sum_{n=1}^\infty \hat{\mu}(-n)z^n$ in $D$, if the measure $\eta$ is defined on $T$ by $\eta(E) = -\int_E \, w d\mu(-w), G = C_\eta$ and thus $G \in H^p(D)$ for each $p < 1$ by Theorem 3.5 of [1].

We can assert a converse to Proposition 2.2. It is an easy consequence of (i) of Theorem 2.4 below that if $p < 1$, and $\mu$ is a measure on $T$, $\tilde{C}_\mu \in L^p_0$ if and only if $\mu$ is a singular measure plus a constant multiple of Lebesgue measure.

A bounded analytic function $\phi$ defined in $D$ is called *inner* if $|\phi(z)| < 1, z \in D$, and $|\phi(e^{i\theta})| = 1$, a.e. $e^{i\theta} \in T$.

**Proposition 2.3.** Let $f \in L^p_0$. If $\phi$ is inner with $\phi(0) = 0$, then $f \circ \phi \in L^p_0$.

**Proof.** Let $f \in L^p_0$. Then there are $G$ and $K$ in $H^p(D)$ with $\tilde{G} = f$ and $\tilde{H} = \tilde{f}$ in $L^p$. That $G \circ \phi$ and $K \circ \phi$ are in $H^p(D)$ follows from Theorem 1.7 of [1]. Let $X$ be the set of $e^{i\theta} \in T$ where $\lim_{r \to 1} H(re^{i\theta}) = \lim_{r \to 1} G(re^{i\theta})$. Since
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$X$ has measure $2\pi$, $\tilde{\phi}(e^{i\theta})$ must be in $X$ for a.e. $e^{i\theta} \in \mathbb{T}$. Thus $(G \circ \phi)^\sim = f \circ \tilde{\phi}$ and $(H \circ \phi)^\sim = f \circ \tilde{\phi} = (f \circ \tilde{\phi})^\sim$, which shows that $f \circ \tilde{\phi} \in L^p_0$.

One more definition before we state the main result of this section. If $F(z) = \Sigma_{n=0}^\infty a_n z^n$ is analytic in $D$, we define $\text{spec } F$ to be $\{n: a_n \neq 0\}$.

**Theorem 2.4.** Let $0 < p < 1$. Suppose that $\mu$ is a finite Borel measure on $\mathbb{T}$.

(i) If $\mu$ is absolutely continuous, then $L^p(\tilde{C}_\mu) = L^p(\text{spec } C_\mu)$.

(ii) If $\mu$ is singular, then $L^p(\tilde{C}_\mu)$ contains no nonconstant exponential functions.

**Proof.** (The equality is clear if $\tilde{C} \in L^1$. But we only have that $\tilde{C}_\mu \in L^r$ for each $r < 1$.) $\tilde{C}_\mu \in L^p(\text{spec } C_\mu)$ since the Fourier series of $\tilde{C}_\mu$ is Abel summable to $\tilde{C}_\mu$ in $\| \cdot \|_p$ (see p. 284 of [5]). Thus, $L^p(\tilde{C}_\mu) \subseteq L^p(\text{spec } C_\mu)$. That $L^p(\text{spec } C_\mu) \subseteq L^p(\tilde{C}_\mu)$ follows by an appropriate adaptation of the discussion on p. 263 of [5]. (ii) follows from Proposition 2.1 and 2.2.

To see that (i) of Theorem 1 is a consequence of Theorem 2.4, take $L^p(\tilde{C}_\mu)$, where $\mu$ is any singular measure on $\mathbb{T}$ with $\int d\mu = 0$.

Theorem 2.4 lends weight to the following conjecture: If $X$ is a measure on $\mathbb{T}$ with absolutely continuous part $\mu$, then $L^p(X)$ and $L^p(\tilde{C}_\mu)$ contain the same exponentials if $p < 1$.

There are other natural topologies besides the norm topology for $H^p$ in the case $0 < p < 1$, in particular, the weak topology and the topology induced by the containing space in the sense of [2]. Routine arguments show that in these topologies the $\mathbb{T}$-invariant subspaces of $H^p$ are in 1-1 correspondence with the subsets of the nonnegative integers, as is the case when $1 < p < \infty$ and $H^p$ has the norm topology.

3. Distinct sets of exponentials spanning the same subspace of $L^p$. If $\mu$ is a finite Borel measure on $\mathbb{T}$, its spectrum is $\{n: \hat{\mu}(n) \neq 0\}$. The following implies (ii) of Theorem 1.

**Theorem 3.1.** Let $0 < p < 1$. Suppose that $\Gamma$ is the spectrum of a singular measure on $\mathbb{T}$ and that $\Delta$ is obtained from $\Gamma$ by deleting a finite number of elements. Then $L^p(\Gamma) = L^p(\Delta)$.

**Proof.** For $0 < r < 1$, define $F_r$ on $\mathbb{T}$ by $F_r(e^{i\theta}) = \Sigma_{n=-\infty}^\infty \hat{\mu}(n)r^n e^{in\theta}$. Since $F_r$ is the $r$th Abel mean of the Fourier series of $\mu$ and $\mu$ is singular, Theorem 1.2 of [1] shows that $\lim_{r \to 1} F_r(e^{i\theta}) = 0$, a.e. $e^{i\theta} \in \mathbb{T}$. $\{F_r: 0 < r < 1\}$ is bounded in $L^1$ and thus $\lim_{r \to 1} \|F_r\|_p = 0$. Let $\Lambda = \{n: n \in \Gamma, n \notin \Delta\}$ and define the trigonometric polynomial $P$ on $\mathbb{T}$ by

$$P(e^{i\theta}) = - \sum_{n \in \Lambda} \hat{\mu}(n)e^{in\theta}.$$
so

\[ \lim_{r \to 1} \left\| \sum_{n \in \Delta} \hat{\mu}(n) r^n e^{in \cdot} - P \right\|_p = 0. \]

Thus \( P \in L^p(\Delta) \), and as a consequence each \( e^{im \cdot} \) with \( m \in \Lambda \) is in \( L^p(\Delta) \), so \( L^p(\Gamma) = L^p(\Delta) \).

**BIBLIOGRAPHY**


DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CALIFORNIA 94305

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