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ON L^1 CONVERGENCE OF CERTAIN COSINE SUMS

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Abstract. It is shown that to a certain cosine series f, a Rees-Stanojević cosine sum g_n can be associated such that g_n converges to f pointwise, and a necessary and sufficient condition for L^1 convergence of g_n to f is given. As a corollary to that result we have a generalization of the classical result of this kind. Other corollaries are given concerning the well-known integrability conditions.

This paper gives an analogue for modified cosine sums of the classical result concerning L^1 convergence of a Fourier sine series. Rees and Stanojević [1] introduced these cosine sums that approximate their pointwise limit "better" than the classical cosine series since they converge in the L^1 metric space to their limit when the classical cosine series may not.

LEMMA 1. *Let $f(x) = \lim_{n\to\infty} f_n(x)$ where $f_n(x) = \tfrac{1}{2}a(0) + \Sigma_{k=1}^n a(k)\cos kx$, $\lim_{n\to\infty} a(n) = 0$, and $\Sigma_{k=0}^\infty |\Delta a(k)| < \infty$. Let $g_n(x) = \tfrac{1}{2}\Sigma_{k=0}^n \Delta a(k) + \Sigma_{k=1}^n \Sigma_{j=k}^n \Delta a(j)\cos kx$. Then $\lim_{n\to\infty} g_n(x) = f(x)$.*

THEOREM 1. *Let f, f_n, and g_n be as defined in Lemma 1. Then g_n converges to f in the L^1 metric if and only if given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\int_0^\delta |\Sigma_{k=n+1}^\infty \Delta a(k) D_k(x)| < \epsilon$ for all $n \geqslant 0$, where $D_k(x)$ is the Dirichlet kernel.*

COROLLARY 1. *Let f_n and f be as defined in Lemma 1. If for $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\int_0^\delta |\Sigma_{k=n}^\infty \Delta a(k) D_k(x)| < \epsilon$ for all $n \geqslant 0$ then f_n converges to f in the L^1 metric if and only if $\lim_{n\to\infty} a(n)\log n = 0$.*

COROLLARY 2. *Let f and g_n be as defined in Lemma 1. If $\Sigma_{n=1}^\infty |\Delta^2 a(n)|(n+1) < \infty$, then g_n converges to f in the L^1 metric.*

COROLLARY 3. *Let f and g_n be as defined in Lemma 1. If $\Sigma_{k=1}^\infty |\Delta a(k)|\log k < \infty$, then g_n converges to f in the L^1 metric.*

COROLLARY 4. *Let f and g_n be as defined in Lemma 1. If $a(n) = b(n) + c(n)$ where $\lim_{n\to\infty} b(n) = \lim_{n\to\infty} c(n) = 0$, $\Sigma_{n=1}^\infty |\Delta b(n)|\log n < \infty$, and $\Sigma_{n=1}^\infty |\Delta^2 c(n)|(n+1) < \infty$, then g_n converges to f in the L^1 metric.*

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[1]Portions of these results appear in a doctoral thesis of John W. Garrett at the University of Missouri-Rolla in 1974.

COROLLARY 5. *Let f and g_n be as defined in Lemma* 1. *If* $a(n) = \alpha(n)\beta(n)$ *where* $\sum_{n=1}^{\infty} |\Delta\alpha(n)| < \infty$, $|\beta(n)| \leqslant M$, $\sum_{n=1}^{\infty} |\Delta^2\beta(n)|(n+1) < \infty$, *and* $\sum_{n=1}^{\infty} |\beta(n)\Delta\alpha(n)| \log n < \infty$, *then g_n converges to f in the L^1 metric.*

Proofs and details of these results will appear elsewhere.

REFERENCE

1. C. S. Rees and Č. V. Stanojević, *Necessary and sufficient conditions for integrability of certain cosine sums,* J. Math. Anal. Appl. **43** (1973), 579–586. MR **48** #794.

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