BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 1, January 1976

CONFORMAL MAPS ON HILBERT SPACE

BY MELVYN HUFF

Communicated February 15, 1975

1. **Introduction.** In [1] Nevanlinna gave a simple proof of the following theorem of Liouville. (Precise definitions appear below.)

THEOREM 1. *Suppose U is a connected open set in a real Hilbert space H of dimension $\geqslant 3$ (including ∞) and $f: U \longrightarrow H$ is C^4 and conformal. Then f is either*

(a) *an affine map whose linear part is a constant multiple of a unitary operator,*

(b) *an inversion with respect to a sphere,*

(c) $f_1 \circ f_2$ *where* f_1 *is of type* (a) *and* f_2 *is of type* (b).

REMARKS. (i) The dimension of H must be $\geqslant 3$ because every holomorphic map on **C** with a nowhere zero derivative is conformal.

(ii) For $\mathbf{R}^n$, the theorem is known even for f just C^1 [2].

(iii) The proof of Nevanlinna depends on f being C^4.

In this paper we outline how a technique in [3], when recognized as applying to conformal mappings and suitably modified, can be used to prove the theorem with f only C^3.

2. **Notation and definitions.** H will be a real infinite dimensional Hilbert space and U a connected open subset. A map is C^n if it is n times continuously Fréchet differentiable as in [4]. A C^1 function $f: U \longrightarrow H$ is called conformal if Df_x is a linear isomorphism and there is a function $c: U \longrightarrow \mathbf{R}$ such that

$$\langle Df_x(h_1), Df_x(h_2)\rangle = c(x)\,\langle h_1, h_2\rangle$$

for all x in U and all h_1, h_2 in H. (This definition is merely a reformulation of the more geometric definition that says f preserves the angle between two curves meeting at a point.) Banach and Hilbert manifolds are defined as in [4].

By an inversion with respect to the sphere $\{x \in H: \|x - p\| = r\}$ I mean the map $x \longrightarrow x'$ where

(i) $\|x - p\|\,\|x' - p\| = r^2$ and

(ii) x and x' lie on the same ray originating at p. The analytic form of such an inversion is

$$x \longrightarrow r^2(x - p)\|x - p\|^{-2} + p.$$

AMS (MOS) subject classifications (1970). Primary 58B10, 58B20, 46C10.

3. **Outline of the proof.** (1) We develop the theory of connections for Banach manifolds and specialize to the case of Riemannian connections for a C^3 Hilbert manifold. For each chart $(W, \psi\colon W \longrightarrow H)$ of the manifold a C^1 function Γ, called the Christoffel function, is defined on $\psi(W)$ such that $\Gamma(y)$ is a continuous H-valued bilinear map on H for each y in $\psi(W)$. The collection of such Γ (together with a coherence property on the overlap of charts) determines and is determined by the connection.

(2) Let $d(x) = 1/\sqrt{c(x)}$. (Since Df_x is one-one, $c(x)$ is not zero.) For example if f is the affine map $f(x) = rL_0(x) + h_0$ where r is real, L_0 unitary and $h_0 \in H$, we have $d(x) = 1/r$. On the other hand for the inversion

$$f(x) = r^2(x-p)\|x-p\|^{-2} + p$$

we have $d(x) = \langle x-p, x-p\rangle/r^2$.

Since Hilbert space with the inner product as Riemannian metric has zero curvature we get the following equation for d:

$$(*) \qquad 2D^2d_x(h_1, h_2) = 2Dd_x(h_1)Dd_x(h_2)/d(x) + Dd_x[\Gamma_x(h_1, h_2)].$$

To derive this we use the fact that the dimension of H is $\geqslant 3$.

(3) We prove that in a neighborhood of each point x_0, the above equation has a unique solution

$$(**) \qquad d(x) = A\langle x-x_0, x-x_0\rangle + \langle b, x-x_0\rangle + C$$

where $C = d(x_0)$, b is the element in H corresponding to Dd_{x_0} under the canonical isomorphism of H with its dual H^* and $A = \langle b, b\rangle/4C$.

The method of proof is to start at x_0 where $(**)$ is true and then to show that equality continues as we move in any direction. Pick a unit vector u and define $g_1(t) = d(x_0 + tu)$. Using $(*)$ the function $K_1(t) = [g_1(t), Dg_1(t)] \in R \times H^*$ is shown to satisfy a differential equation of the form $K'(t) = F[t, K(t)]$ with initial condition $K(0) = [d(x_0), Dd_{x_0}]$. Letting $g_2(t) = A\langle tu, tu\rangle + \langle b, tu\rangle + C$ we verify that $K_2(t) = [g_2(t), Dg_2(t)]$ satisfies the same differential equation and initial condition. The equality of g_1 and g_2 follows from uniqueness.

(4) Using the connectedness of U we get that the local solution in (3) is actually a global solution for d.

(5) We show that if $f\colon U \longrightarrow H$ and $g\colon U \longrightarrow H$ are C^3 conformal maps such that g is one-one and $d_f = d_g$ (where d_f is the d corresponding to f), then there is a vector h in H and unitary operator L such that $f = L \circ g + h$.

(6) From (4) we know that $d(x) = A\langle x-x_0, x-x_0\rangle + \langle b, x-x_0\rangle + C$.

Case 1. $b = 0$ and $A = \langle b, b\rangle/4C = 0$ in which case $d(x) = C$ has the same form as the d for an affine map as in (2), if $C = 1/r$.

Case 2. $b \neq 0$ and thus $A \neq 0$. Then $d(x) = \langle x-p_0, x-p_0\rangle/r$ where $r = 4C/\langle b, b\rangle$ and $p = x_0 + 2Cb/\langle b, b\rangle$. This is the same form as the d for an inversion as in (2) above.

(7) Combining (6) with (5) we get our theorem.

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DEPARTMENT OF MATHEMATICS, TUFTS UNIVERSITY, MEDFORD, MASSACHUSETTS 02155

Current address: 329A Summit Ave., Brookline, Massachusetts