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BOOK REVIEWS

Algebraic topology—homotopy and homology, by Robert M. Switzer, Die Grundlehren der math. Wissenschaften, Band 212, Springer-Verlag, Berlin, 1975, xiv+526 pp., $52.50.

This is a very advanced algebraic topology text. Assuming a reader with at least a year's background in algebraic topology, the author proposes to develop the fundamentals of stable homotopy theory, so that by the end of the book the reader would be ready to do research.

In order to try to describe stable homotopy theory, one may begin with a rather obvious question. Being given two spaces X and Y, one would like to describe the set of maps from X to Y, i.e., the continuous functions $f:X\to Y$. Clearly one has the constant maps ($f(x)=y_0$ for all $x\in X$) and fixing a point $x_0\in X$ one can assign to each map f the point of Y given by $f(x_0)$. An immediate, and essential, simplification is then to fix points $x_0\in X$ and $y_0\in Y$, and ask for a description of the set of maps $f:X\to Y$ with $f(x_0)=y_0$.

The question is far too general for a reasonable solution, and the set of maps is frequently so large as to be unmanageable. Homotopy theory approaches this question by introducing an equivalence relation on maps. Given two maps $f_0, f_1:X\to Y$ ($f_i(x_0)=y_0$), one says f_0 and f_1 are homotopic if there are maps $f_t:X\to Y$, $0\leqq t\leqq 1$, with $f_t(x_0)=y_0$, so that the resulting function $F:X\times[0,1]\to Y:(x,t)\to f_t(x)$ is continuous. The set of equivalence classes, denoted $[X;Y]$ suppressing the chosen points from the notation, is called the set of homotopy classes of maps from X to Y. The hope is that $[X;Y]$ captures much of the structure of the set of maps while at the same time being smaller and more tractible, and this is frequently the case.

If Y is a topological group with y_0 the unit, multiplication of functions makes $[X;Y]$ into a group. One can also define a group structure on $[X;Y]$ if X is suspension, i.e., if there is a space X', $x_0'\in X'$, for which X is the quotient space of $X'\times[0,1]$ obtained by identifying $X'\times\{0\}\cup X'\times\{1\}\cup\{x_0'\}\times[0,1]$ to a point x_0. One writes $X=\Sigma X'$. If X' is also a suspension, then for each Y, $[X;Y]$ can be given the structure of an abelian group, and it could be studied and described as an algebraic object.

Being given a map $f:X\to Y$ ($f(x_0)=y_0$ as always) the function $f\times\text{identity}:X\times[0,1]\to Y\times[0,1]$ induces a map $\Sigma f:\Sigma X\to\Sigma Y$, giving a function $\Sigma:[X;Y]\to[\Sigma X;\Sigma Y]$. One may, of course, iterate this construction, and one lets $\{X;Y\}$ be the limit of the sets $[\Sigma^r X;\Sigma^r Y]$. This is the set of stable homotopy classes of (stable) maps of X into Y. It is always an abelian group and is quite manageable, although by no means computable. For example, a major open question is to compute these groups when X and Y are spheres, S^{n+k} and S^n respectively.

In attempting to study maps from X to Y one is then led to consider not just spaces, but sequences of spaces: X, ΣX, $\Sigma^2 X, \ldots$. Expanding one's horizons, one defines a spectrum $\boldsymbol{E}$ to be a sequence of spaces $\{E_i\}$ together with maps $e_i : \Sigma E_i \rightarrow E_{i+1}$. Recently Michael Boardman has given an adequate definition of a map between spectra so that nice properties follow. This definition has been reformulated by J. Frank Adams, and is presented in the book. The precise description is too technical to present here, but vaguely a map $\varphi : \boldsymbol{E} \rightarrow \boldsymbol{F}$ is a collection of maps $\varphi_i : E_i' \rightarrow F_i$ with $\varphi_{i+1} \circ \Sigma e_i = f_{i+1} \circ \varphi_i$, where $E_i' \subset E_i$ are subspaces forming a subspectrum of $\boldsymbol{E}$ which as i gets large become most of E_i (the precise sense of "most" being the crucial technical point).

Stable homotopy theory is then the study of spectra and the homotopy classes of maps between them. Its goal is to help understand the set of maps from X to Y. The material in this book and in the subject is the development of the tools and techniques for a successful and efficient calculation of the group of homotopy classes of maps between two spectra.

With this rather limited description of stable homotopy, I would encourage the general reader to turn to the next review. The book is suitable only for a specialist or potential specialist, and I will now address myself to such a likely reader or buyer.

A topologist who examines this book is immediately struck by several of its features. It is 500 pages long, has a shocking price tag, and appears to cover a tremendous amount of recent algebraic topology. It is unfortunate that the author has tried to cover so much, for by doing so, the book is probably not worth the price to most readers.

At least a third of the book is standard elementary homotopy theory and homology-cohomology theory, and the presentation is standard. I would think it easier to learn this material from the books by Hu or Spanier which provide broader coverage and seem to me to give better insight for a beginning student.

The material on examples of generalized homology and cohomology theories (vector bundles, K-theory, manifolds, orientation and duality, and characteristic classes) is extremely abbreviated. A reader wanting to learn this material would have to go elsewhere to develop any real familiarity with these subjects.

The best part of the book, nearly half, is genuinely stable homotopy—the author's real interest. It is suitable for a reader familiar with the above material, which would have been a better choice of an audience. The category of spectra is constructed, representation theorems for generalized homology and cohomology are proved, a very comprehensive treatment of products is given, and cohomology operations and (dually) homology cooperations are discussed. The culmination is to set up the Adams spectral sequence, in homology as J. Frank Adams now tells us is the right way, and to illustrate its use by calculating some cobordism theories.

Ignoring occasional misprints, I found a few errors, but surprisingly few. The definition of a contractible space on p. 36 is not the standard one and is not always the same. The discussion of lens spaces S^3/Z_p on p. 90 is incorrect, for the integral and Z_p cohomology rings are in fact independent of the Z_p action. The homotopy types can be distinguished only by getting chosen generators from the reduction of integral to mod p cohomology, or from the Hurewicz homomorphism. The most crucial mistake is in 14.8 on p. 311, where the author says that $\tilde{E}^0(B)=0$ if $\boldsymbol{E}$ is a connected spectrum when in fact one needs $\pi_q(E)=0$ for $q>0$. This error is compounded in the following discussions of orientation. In particular, 14.9 is false except for ordinary cohomology and the proof of 14.18 is only valid then. The exercise on p. 312 suffers badly from misprints but seems to involve the same error, and if I correctly interpret it, it is false.

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Almost periodic differential equations, by A. M. Fink, Lecture Notes in Mathematics, No. 377. Springer-Verlag, Berlin, Heidelberg, New York, 1974, vii+336 pp.

Nonlinear differential equations of higher order, by R. Reissig, G. Sansone and R. Conti, Noordhoff, Leyden, 1974, xiii+669 pp.

Functional differential equations, by J. K. Hale, Applied Mathematical Sciences, No. 3. Springer-Verlag, New York, Heidelberg, Berlin, 1971, viii+238 pp.

In this review we trace some of the major developments in the study of the qualitative behavior of solutions of ordinary differential equations and show how these books fit into this general theory.

I. **Origins of the qualitative theory.** The qualitative theory of ordinary differential equations began nearly a century ago with the work of H. Poincaré in France and A. Lyapunov in Russia. Prior to this time the major emphasis in differential equations had been on the methods of "solving" various equations either in closed form by an explicit formulation, or in terms of series, cf. Ince [**17,** pp. 529–539] for example. This interest in solving equations was undoubtedly influenced by the strong interconnection between the study of differential equations and the problems of physics. To put it in modern language, the existence of a solution is clearly the first logical step in establishing the validity of a given mathematical model for a physical phenomenon. Naturally, the first attempts at finding solutions were in terms of explicit formulae. This line of research reached its dénouement during the period from 1875 to 1900 when the work of Lipshitz, Picard, Peano, and others established the so-called fundamental theory, i.e., the theory of the existence, uniqueness and continuity of solutions. While investigations into the fundamental theory continue even today, one finds that the major emphasis in the study of ordinary differential equations now seems to be in the qualitative behavior of solutions.