25. J. von Neumann, *Almost periodic functions in a group*. I, Trans. Amer. Math. Soc. **36** (1934), 445–492.

26. L. S. Pontrjagin, *Continuous groups*, GITTL, Moscow, 1938; English transl., *Topological groups*, Princeton Math. Ser., no. 2, Princeton Univ. Press, Princeton, N.J., 1939. MR **1**, 44.

27. R. Reissig, G. Sansone and R. Conti, *Qualitative Theorie nichtlinearer Differentialgleichungen*, Edizioni Cremonese, Rome, 1963. MR **28** #1347.

28. G. R. Sell, *Topological dynamics and ordinary differential equations*, Van Nostrand Reinhold, London, 1971.

29. ———, *Topological dynamical techniques for differential and integral equations*, Ordinary Differental Equations, Academic Press, New York, 1972, pp. 287–304.

GEORGE R. SELL

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 2, March 1976

Fourier analysis on local fields, by M. H. Taibleson, Mathematical Notes, Princeton University Press, Princeton, New Jersey, 1975, xii+294 pp., $7.00.

This book contains the lecture notes of a course given by the author at Washington University, Saint Louis during the Fall and Spring semester 1972–1973. Many results have appeared earlier in a series of papers, some of them written in collaboration with P. Sally, R. A. Hunt and K. Phillips. We find in this book well-known concepts from classical analysis: the Fourier transform, the Hankel transform, Gamma, Beta and Bessel functions, the Poisson summation formula, Fourier series, Césaro sums, fractional integration and many others. But from the title of the book it is clear that these subjects are treated here for a situation different from the classical one. In the classical case these subjects are discussed in the context of analysis on a euclidean space. In this book, however, the theory is developed for local fields, with emphasis, almost exclusively, on totally disconnected fields and on the analogy between this case and the euclidean case.

It is striking to observe the enormous evolution of the subject in two centuries, especially the revolution in the last fifty years. The great lines of this development are most interesting and they are an excellent illustration of the influence of algebra and topology on the form and contents of contemporary analysis.

Fourier series were studied by D. Bernoulli, D'Alembert, Lagrange and Euler from about 1740 onwards. They were led by problems in mathematical physics to study the possibility of representing a more or less arbitrary function f with period 2π as the sum of a trigonometric series of the form

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx).$$

Dirichlet (1829) and later Riemann (1854) started the study of these series in a more rigorous way. This was continued by Cantor—who showed that a representation at *all* points by a convergent trigonometric series is necessarily unique if it exists at all (1870)—by Du Bois Reymond (1872), Lebesgue (about 1905) and several other mathematicians. It appeared that there are continuous periodic functions that admit no such representations. These studies led to a more general representation theory, replacing the condition of convergence at all points by the weaker condition of convergence at almost all points, and the corresponding uniqueness problem. This period may be described as the classical stage (but these problems are studied up to our days).

A second period started in the twenties with the application of the methods of functional analysis, a discipline that began its important course in those years (although the origins, to be found in the work of Volterra, go back to the end of the 19th century) culminating in those years in Banach's *Théorie des opérations linéaires* (1932). Banach and Steinhaus, for instance, proved (1927) again, now with the methods of functional analysis, the existence of a continuous function whose Fourier series is divergent at a special point and they improved the result by means of the principle of condensation of singularities that goes back to Hankel (1870). In the classical book *Trigonometrical series*, Zygmund (1935) used the results of the young functional analysis. It is curious to mention this book because it is characteristic for the rapid growth of the subject that the second edition, published in 1968, has two volumes each of which has about 380 pp., while the first edition alone has about 320 pp. Nowadays one cannot do without functional analytic methods in this domain.

But this is not the essential point in the presentation of the subject in Taibleson's book because functional analysis is nowadays applied in most parts of analysis. The crucial point seems to be the influence of modern algebraic methods and this enlarges the scope of the theory in an important way, adding a new dimension to it. It may be stated that with the application of algebra a third phase was started.

It is evidently not the place here to write a survey of the gradual application of the concepts of what was formerly called "modern algebra" to analysis. "Modern algebra" developed gradually from abstract group theory at the second half of the 19th century into the theory of rings and fields and this development is closely connected with the work of Emmy Noether, Artin, and Van der Waerden in the twenties. This was in about the same period as functional analysis started growing. In those years topological results also entered the picture, leading to the concept of a topological group. Some aspects of this development are necessary for understanding Fourier analysis on local fields.

Of the utmost importance was a result, proved by Haar in 1933, stating that on any locally compact topological group there exists an invariant measure, the Haar measure. Already in 1940 Weil applied this result to representation theory and Fourier analysis on commutative locally compact groups. In his book *L'intégration dans les groupes topologiques et ses applications*, one finds a treatment of the Fourier transform on groups and its properties, for instance the formula of Plancherel, etc. Haar measure also is the essential tool for Fourier analysis on local fields. Before describing the topics treated by Taibleson in his book, some other algebro-topological facts must be mentioned.

The very beginning of the theory is the introduction of p-adic numbers and the p-adic fields in 1908 by Hensel. At first they were mainly studied in number theory and algebra. Their introduction is usually connected with the name of Hensel. But, as Weil observes in his introduction to Kummer's *Collected works*, the idea goes back to Kummer from about 1850 onwards. He realized that formal power series can be used to define a number modulo p^n for arbitrary high values of n. In 1912 Kürschak placed Hensel's p-adic fields in a more general frame: in a paper *Limesbildung und allgemeine Körpertheorie* he defined the concepts of a valuation and valued fields. In 1918 Ostrowski continued this study in a paper *Über einige Lösungen der Funktionalgleichung* $\phi(x)\cdot\phi(y)=\phi(xy)$. He introduced the nonarchimedean valuations, characterized by the strong triangle inequality

$$|x+y|\leq\max(|x|,|y|).$$

From their work it became clear that next to the real number field, as the completion of the rational number field with respect to the ordinary absolute value, there exist infinitely many other complete valued fields which are of equal rank in a natural way. They are the p-adic fields $\boldsymbol{Q}_p$, obtained by the completion of $\boldsymbol{Q}$ with respect to the p-adic valuations. These fields are locally compact and totally disconnected. These developments led to the more general concept of a *local field*, which is at the basis of Taibleson's book. His apparatus for Fourier analysis are the methods of functional analysis and the algebraic and topological properties of local fields. Taibleson gives the following definition: *a local field K is a locally compact, nondiscrete, totally disconnected field.* The first chapter of the book under review contains an introduction to the properties of local fields which are basic for Fourier analysis. Any local field is a valued field and the valuation satisfies the inequality (called the ultrametric inequality)

$$|x+y|\leq\max(|x|,|y|).$$

Analysis, in particular Fourier analysis, over a local field K can be studied in two ways. On the one hand one can study functions defined on K which take their values in K. In this direction functional analysis was studied from about 1940 onwards (for the results in this direction see Monna [**4**], containing a bibliography until 1970). Schikhof [**5**] obtained some results on Fourier analysis with values in K for suitable groups (there are difficult problems in the general case).

On the other hand one can study real- or complex-valued functions defined on K. This is the case treated by Taibleson. All functions and characters on the group are complex valued. Results on Fourier analysis for local fields were already obtained by Tate in 1950 in his Thesis *Fourier* *analysis in number fields and Hecke's zeta-functions.* This thesis remained unpublished for many years, but it had a deep influence on the subject as a piece of clandestine literature. It was published for the first time in the Proceedings of the Brighton Conference held in 1965; see [**1**]. Further results on representations of SL(2, K) and the special functions on local fields were published in a fundamental paper of Gelfand and Graev (1963; [**3**]). The interest of Gelfand and Graev in special functions on K came from the representation theory of SL(2, K) in which these functions play a role. For SL(2, $\boldsymbol{R}$) this was already known, but for the generalization to SL(2, K) they needed the special functions on K and therefore they developed the theory. On the basis of these papers Sally and Taibleson set out a program of developing and extending the basic facts about harmonic analysis on local fields and the n-dimensional vector spaces over the fields, with an emphasis on the analogy between the local field theory and the euclidean case. This program is carried out and described in the book under review.

The book is divided into eight chapters each of which contains several sections. The chapter titles and the section headings give an indication of the contents. We add some remarks which may be useful in comparing the euclidean case and the case of local fields.

Chapter I. Introduction to local fields. 1. Rademacher functions and the Walsh-Paley group. 2. The 2-adic numbers and 2-series numbers. 3. p-adic and p-series numbers. 4. Classification of local fields. 5. Properties of local fields. 6. More facts about K. 7. The dual of K^*. 8. The dual of K^+. 9. Notes for Chapter I.

In this chapter a Haar measure dx is introduced for the locally compact additive group K^+ of K. A Haar measure on the locally compact multiplicative group K^* is then $dx/|x|$. Furthermore the continuous multiplicative characters π on K^* are defined, that is $\pi \in \hat{K}^*$; in general they are not assumed to be unitary. If π is such a character then $\pi = \pi^*|\cdot|^\beta$, where $\beta \in \boldsymbol{C}$ and π^* is a unitary character on O^*, O^* being the group of units in K^*, i.e. $O^* = \{x \in K \mid |x| = 1\}$. The existence of nontrivial, additive, unitary, continuous characters χ on K^+ follows from the Pontryagin Duality Theorem. These two types of functions play a fundamental role in the following chapters: the multiplicative characters are generalizations of a power function, the additive characters are generalizations of the exponential function. It is the interaction between additive and multiplicative structures that should be kept in mind when comparing Fourier analysis in the euclidean case with the theory on local fields.

Chapter II. Fourier analysis on K, the one-dimensional case. 1. The L^1 theory. 2. The L^2 theory. 3. Distributions on K. 4. The Mellin transform: Fourier analysis on K^*. 5. Gamma, Beta and Bessel functions. The Hankel transform. 6. Some elementary aspects of Fourier analysis on the ring of integers on a local field. 7. Notes for Chapter II.

If $f \in L^1$ the *Fourier transform* of f is the function

$$\hat{f}(x) = \int_K f(\xi)\bar{\chi}_x(\xi)\, d\xi = \int_K f(\xi)\chi(-x\xi)\, d\xi.$$

This chapter contains the analogues of the well-known properties of this transform: multiplication theorems, the inversion of the transform, unitary on L^2, the Fourier transform for distributions (the test functions are defined in a suitable way). In this chapter the special functions are defined and some of their properties are proved.

The *Gamma function* $\Gamma(\pi)=\Gamma(\pi^*|\cdot|^\alpha)=\Gamma_{\pi^*}(\alpha)$ is defined for all (not necessarily unitary) characters π, except for the identity character. If π is ramified

$$\Gamma(\pi) = \Gamma_{\pi^*}(\alpha) = P\int \bar{\chi}(x)\pi(x)\,|x|^{-1}\, dx,$$

where the principal value $P\int$ is defined in a suitable way. There is also a definition for the case that π is unramified; we omit it here.

The *Beta function* is defined as follows. Let $\pi=\pi^*|\cdot|^\alpha$, $\lambda=\lambda^*|\cdot|^\beta$ be multiplicative characters. Then

$$B(\pi, \lambda) = \frac{\Gamma(\pi)\Gamma(\lambda)}{\Gamma(\pi\lambda)} = \frac{\Gamma_{\pi^*}(\alpha)\Gamma_{\lambda^*}(\beta)}{\Gamma_{\pi^*\lambda^*}(\alpha+\beta)}.$$

$B(\pi, \lambda)$ can be considered as a meromorphic function of the two complex variables α, β for π^*, λ^* fixed. It is proved that B can be represented in the form of an integral.

For $\pi \in {}^{-}\hat{K}^*$; u, $v \in K^*$, the *Bessel function* of order π, denoted $J_\pi(u, v)$, is the principal value integral

$$P\int \bar{\chi}(ux+(v/x))\pi(x)\,|x|^{-1}\, dx.$$

The author observes that this definition can be extended to obtain Bessel functions of more general order. By means of the Bessel function the Hankel transform of order π ($\pi \in \hat{K}^*$) is defined. The reader should compare these definitions with the usual definitions in complex analysis. The analogue of the classical theory of Fourier series on the torus $\boldsymbol{T}$ is harmonic analysis on the ring of integers O of K. The Fourier coefficients are defined with respect to a complete orthonormal system of additive characters on O. The analogue of the formula of Plancherel is proved.

Chapter III. Fourier analysis on K^n. 1. The L^1 theory. 2. The L^2 theory. 3. Distributions on K^n. 4. Riesz fractional integration. 5. Bessel potentials. 6. The relation of I^α and J^α. 7. Lebesgue spaces of Bessel potentials. 8. An elementary aspect of the Lipschitz theory. 9. Duals of the L^p_α spaces. 10. Operators that commute with translations. 11. Notes for Chapter III.

The results of the preceding chapters are extended to the n-dimensional case. The norm on K^n is defined by

$$|(x_1, \ldots, x_n)| = |x| = \sup_i |x_i|.$$

Chapter IV. Regularization and the theory of regular and subregular functions. 1. Regular functions on $K^n \times \mathbf{Z}$. 2. More about Lipschitz spaces. 3. Subregular functions, domains, regular majorants and nontangential convergence. 4. Notes for Chapter IV.

Chapter V. The Littlewood-Paley function and some applications. 1. The Littlewood-Paley function. 2. Local equivalence of nontangential convergence, nontangential boundedness and the existence of Sf. 3. Notes for Chapter V.

Chapter VI. Multipliers and singular operators. 1. Multipliers. 2. Special cases of the multiplier theorem. 3. Application of the multiplier theorem to Fourier series. 4. Singular integral operators. 5. Notes for Chapter VI.

Chapters IV, V and VI are of a somewhat different kind. Using results from the preceding chapters the author studies functions called regular. They are defined on $K^n \times \mathbf{Z}$ and are the analogues of the harmonic functions on $\boldsymbol{R}^n \times (0, \infty)$, that is the euclidean upper half space. Regular functions are defined by means of a smoothness condition and a substitute for the mean value property of harmonic functions. A maximal principle for regular functions is proved. The concept of the regularization of a distribution on K^n is defined. It is proved that it is a regular function on $K^n \times \mathbf{Z}$, satisfying some conditions, and conversely. Further properties of the regularization are proved. Furthermore there are properties of subregular functions, which are the analogues of subharmonic functions, a study of the "boundary behaviour" of regular functions (for boundary questions special definitions are necessary because K is totally disconnected). For the euclidean case see [**6**]. Chapter VI contains a survey of the theory of L^p-multipliers on K^n. In general the multiplier problem can be defined as follows. Given two sets F and G of distributions, it is required to find necessary and sufficient conditions on the complex-valued function Φ in ordèr that

$$f \in F \Rightarrow \Phi \hat{f} \in \mathscr{F}G,$$

where $\mathscr{F}G$ denotes the set of Fourier transforms $\hat{g}$ of the elements of g of G. The theory of multipliers is applied to Fourier series in the n.a. case. For the euclidean case see [**2**].

Chapter VII. Conjugate systems of regular functions and an F. and M. Riesz theorem. 1. A fundamental lemma. 2. Construction of conjugate systems on $K^n \times \mathbf{Z}$. 3. F. and M. Riesz theorem. 4. Notes for Chapter VII.

In this chapter conjugate systems of functions defined on a domain $D \subset K^n \times \mathbf{Z}$ are studied. They should be considered as a generalization of pairs of harmonic functions defined by the Cauchy-Riemann differential equations. The system is defined by means of certain linear difference equations. Examples are given. The Riesz theorem concerns conditions under which a certain Borel measure μ is absolutely continuous; the condition is expressed in terms of the support of $\hat{\mu}$.

Chapter VIII. Almost everywhere convergence of Fourier series. 1. The results. 2. Notation and preliminary results. 3. Proof of the basic result. A. Reduction to the Basic Lemma. B. Development of the assertions. C. Proof of the Basic Lemma. D. Proof of the assertions. 4. Notes for Chapter VIII.

In this chapter the author treats the local field analogue of the problem of the convergence almost everywhere of the Fourier series of certain classes of functions. The conjecture of Lusin, namely that if $f \in L^2(0, 2\pi)$ its trigonometric Fourier series converges almost everywhere, was proved by Carleson in relatively recent years (1966). Hunt extended this result to L^p $(p>1)$ shortly thereafter. In the local field case the Fourier series of f is given by

$$f(x) \sim \sum_{n=0}^{\infty} C_n \chi_n(x)$$

where (χ_n) is a complete set of additive characters on the ring O of integers. Denote by $S_n(f)$ the nth partial sum of this series. The main result is that when $f \in L^p(O)$,

$$S_n f(x) \to f(x)$$

almost everywhere as $n \to \infty$.

Each of the Notes at the end of the chapters contains supplementary information on the subject treated in that chapter: remarks facilitating the comparison between the euclidean case and the local field theory, references to old and new literature and to results that are not treated in the book under review. Obviously it is not possible to give a detailed review of all the topics that are treated in this book but from this outline of the contents it may be clear that this is a very interesting book. It can be recommended to anyone who is working in this field, but also the nonexpert can profit from it because of the clear presentaion of the subject.

References

1. J. W. S. Cassels and A. Fröhlich, *Algebraic number theory*, Proc. Instructional Conf. organized by the London Math. Soc., London and New York, 1967.

2. R. E. Edwards, *Fourier series. A modern introduction.* Vols. I, II, Holt, Rinehart and Winston, New York, 1967. MR **35** #7062; **36** #5588.

3. I. M. Gel'fand and M. T. Graev, *Representations of the group of second-order matrices with elements in a locally compact field and special functions on locally compact fields*, Uspehi Mat. Nauk **18** (1963), no. 4 (112), 29–99 = Russian Math. Surveys **18** (1963), no. 4, 29–99. MR **27** #5864.

4. A. F. Monna, *Analyse non-archimédienne*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 56, Springer-Verlag, Berlin and New York, 1970. MR **45** #4101.

5. W. H. Schikhof, *Non-archimedean harmonic analysis*, Dissertation, Katholieke Universiteit, Nijmegen, The Netherlands 1967.

6. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Math. Ser., no. 32, Princeton Univ. Press, Princeton, N.J., 1971. MR **46** #4102.

A. F. Monna