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Introduction to the theory of entire functions of several variables, by L. I. Ronkin. Translations of Mathematical Monographs, **44,** Amer. Math. Soc., Providence, Rhode Island, 1974, V+273 pp., $28.50.

In one variable there are a number of excellent textbooks on the theory of entire functions. In several variables, the theory has grown steadily throughout the last 25 years and has gained a respectable size. Therefore the appearance of the English translation of Ronkin's book is highly welcome. The major emphasis is upon the growth behavior of entire functions and upon the construction of these functions with a given growth behavior. The distribution of zeros of entire functions and the construction of these functions with given zeros receive only limited attention. In the selection of the material there is some similarity with Lelong's Montreal Lecture Notes [**2**], but Ronkin's book also brings to us the results of the Russian School. The reviewer's Whitewater Notes [**5**] are a survey of results on the construction of entire functions with given zeros with growth conditions, thus the overlap is minimal. These three books are still the only ones available but in combination they cover most of the essential topics in the theory of entire functions of several variables.

The introduction does not introduce the highlights of the book, which would have been helpful, but assembles a variety of topics and facts useful in the later part of the book. Proofs should have been omitted; this is not the place for them. Chapter I provides an excellent introduction to the theory of subharmonic functions and so does Chapter II on pluri-subharmonic functions. The concept of Γ-capacity is introduced, and certain sets associated to families of pluri-subharmonic functions are shown to be of zero Γ-capacity. Here Lelong uses the concepts of polar and negligible sets instead. Some remarks about the connection between these concepts would have been helpful. The growth behavior of functions of class $\mathfrak{A}$ and $\mathfrak{B}$ are studied. Hopefully someone will invent better names. Denote $\boldsymbol{R}_+=\{x\in\boldsymbol{R}\mid x\geqq 0\}$. Define $\beta_n:\boldsymbol{C}^n\to\boldsymbol{R}_+^n$ by $\beta_n(z_1,\cdots,z_n)=(|z_1|,\cdots,|z_n|)$. Then $\Phi:\boldsymbol{R}_+^n\to\boldsymbol{R}\cup\{-\infty\}$ belongs to $\mathfrak{A}$ if and only if $\Phi\circ\beta_n$ is pluri-subharmonic on $\boldsymbol{C}^n$. A function $\Phi:\boldsymbol{C}^n\times\boldsymbol{R}_+\to\boldsymbol{R}\cup\{-\infty\}$ belongs to $\mathfrak{B}$ if and only if $\Phi\circ(\mathrm{Id}\times\beta_1)$ is pluri-subharmonic and e^Φ is continuous. These classes have fine growth properties and practically all growth measures of entire functions belong to them.

After these two chapters, the quality of the presentation begins to deteriorate. Chapter III deals with the main purpose of the book. The growth behavior of an entire function is described by geometric means. These descriptions are shown to be characteristic. Consider just one of the problems, to get the flavor. Let $f:\boldsymbol{C}^n\to\boldsymbol{C}$ be an entire function. For each $r=(r_1,\cdots,r_n)\in\boldsymbol{R}_+^n$ define

$$M_f(r)=\max\{|f(z_1,\cdots,z_n)|\mid |z_j|=r_j,\ \forall j=1,\cdots,n\}.$$

Then $\log^+M_f$ belongs to class $\mathfrak{A}$. Let $B(f)$ be the interior of the set of all

$a=(a_1, \cdots, a_n)\in \boldsymbol{R}_+^n$ such that a constant r_a exists such that

$$\log^+ M_f(r) \leqq r_1^{a_1} + \cdots + r_n^{a_n}$$

for all $r=(r_1, \cdots, r_n)$ with $|r|>r_0$. If $B(f)\neq\emptyset$, then f is said to have finite order. Let $S(f)=\partial B(f)$ be the boundary of f. Then each $r\in S(f)$ is called a system of associated orders. If x is a vector with positive coordinates in $\boldsymbol{R}^n$, one and only one $\lambda\geqq 0$ exists such that $\lambda x\in S(f)$. The order of f and the jth order of f are defined analytically and denoted by $\operatorname{Ord} f$ and $\operatorname{Ord}_j f$ respectively. Geometrically they are obtained by $(\operatorname{Ord} f, \cdots, \operatorname{Ord} f)\in S(f)$ and by

$$\operatorname{Ord}_j f = \inf\{r_j \mid (r_1, \cdots, r_n)\in B(f)\}.$$

Then

$$\operatorname{Ord}_j f \leqq \operatorname{Ord} f \leqq \operatorname{Ord}_1 f + \cdots + \operatorname{Ord}_n f.$$

A subset B of $\boldsymbol{R}^n$ is called octant-like (the concept cries for a better name) if $x=(x_1, \cdots, x_n)\in B$ and $y=(y_1, \cdots, y_n)$ with $y_j\geqq x_j$ for all $j=1, \cdots, n$ imply $y\in B$. Define $j(x_1, \cdots, x_n)=(1/x_1, \cdots, 1/x_n)$. Let $B\neq\emptyset$ be open in $\boldsymbol{R}^n$ and contained in $\boldsymbol{R}_+^n$. Then $B=B(f)$ for some entire function f if and only if B is octant-like and $j\circ B$ is convex (Theorem 3.1.3). This solves the characterization problem of the possible order behavior of a function of finite order. Similarly, if an associated order r is given, the possible behavior of the associated types of a function of this order r can be characterized. A holomorphic function $f: \boldsymbol{C}^{n+1}\to \boldsymbol{C}$ can be considered as an entire function of its last variable alone depending on n complex parameters. The growth measures are functions of class $\mathfrak{B}$. The generic growth is identified and the deviation from the generic growth is studied.

Subsequently, functions of order 1 and functions of exponential type are investigated. Fourier transforms are considered and the Plancherel-Pólya theorem is proved extending the Paley-Wiener theorem to several variables.

According to Lelong, the radial indicator of an entire function f of positive, finite order ρ on $\boldsymbol{C}^n$ is defined by

$$L_f(z) = \limsup_{t\to\infty} \frac{\log|f(tz)|}{t^\rho} < +\infty$$

for each $z\in\boldsymbol{C}^n$. The regularization L_f^* of L_f is pluri-subharmonic with $L_f^*(tz)=t^\rho L_f^*(z)$ for all $z\in\boldsymbol{C}^n$ and $t>0$. Also L_f is independent of the choice of origin and $\{z\in\boldsymbol{C}^n \mid L_f(z)<L_f^*(z)\}$ is an F_σ-set of Γ-capacity zero. Inversely, if $0<\rho\in\boldsymbol{R}$, if u is pluri-subharmonic on $\boldsymbol{C}^n$ with $u(tz)=t^\rho u(z)$ for all $z\in\boldsymbol{C}^n$ and $t>0$, then an entire function f of order ρ exists such that $L_f^*=u$. This difficult theorem was proved by Kiselman for $\rho=1$ and independently by Martineau in general. Martineau's proof is given and requires the introduction of L^2-estimates on the $\bar\partial$-operator. In §6 of Chapter III, this theory is sketched for $\boldsymbol{C}^n$ closely following the textbook of Hörmander [**1**].

Chapter IV is devoted to the distribution of zeros of entire functions of several variables. After an unsatisfactory auxiliary section on integration over an analytic set, the growth of the zero divisor of an entire function is studied. The canonical function of a divisor of finite order is constructed by Lelong's method and Ronkin's integral representation is given. These parts of Chapter IV have been totally remodeled in [**5**]. At the end an integral representation of an entire function of minimal growth with respect to the last variable is given; the other variables being parameters.

The book covers a wide area, but important parts are missing. Absent are meromorphic functions, the Poisson-Jensen formula, the First and Second Main Theorems of value distribution, and the theory of functions of finite λ-order. After the Russian edition appeared, essential new questions were asked and important new results were obtained. Let $f:\boldsymbol{C}^n \to \boldsymbol{C}^p$ be a holomorphic map such that $A=f^{-1}(0)$ has pure dimension $n-p \geqq 0$. Can the growth of A be estimated by the growth of the coordinate functions of f? This is the Bézout Problem of Griffiths. The answer is negative in general as Cornalba and Shiffman have shown. In some cases, partial answers were given, but the problem is still poorly understood. Skoda [**4**] solved a deep, long-outstanding problem. Let A be an analytic set of pure dimension p in $\boldsymbol{C}^n$; then there exist $n+1$ holomorphic functions $f_1, \cdots, f_{n+1}$ whose growth can be canonically estimated by the growth of A and such that A is the common zero set of $f_1, \cdots, f_{n+1}$. In the unit disc, the Blaschke product is the canonical function of a divisor satisfying a Blaschke condition. It is the analogue of the Weierstrass product without weights. Recently, Skoda and Henkin showed independently, that a divisor satisfying a Blaschke condition in a strictly pseudoconvex domain in $\boldsymbol{C}^n$ is the zero divisor of a holomorphic function in the Nevanlinna class. Of course, these new results could not have been included in the book, but one could have hoped for a question here and there pointing the way to the near future.

The book brings not much new. Little attention has been given to cast it into one mold. Some of the material is taken practically unaltered from the original source, Chapter III, §6 from Hörmander's book (with some alterations and improvements) and almost all of Chapter IV, §§2, 3 from Ronkin [**3**]. The material should have been reorganized around a common theme, the highlights should have been brought out better and their importance should have been expounded. Such a reorganization would have helped the reader to gain a better understanding of the theory and to inspire him to penetrate to new frontiers. For instance, Theorem 4.3.6 on p. 358 is one page long and couched in complicated language. The reader's reaction may well be a feeling of relief that the book will end within five pages.

Occasionally the author is vague, misleading or even wrong. For instance on p. 107 the following is stated "Let $\Phi(r) \in \mathfrak{A}$ be a function of finite order $\rho(\Phi)$. Let $B_\rho = B_\rho(\Phi)$ denote the set of all points $a \in \boldsymbol{R}_+^n$ such that for $|r| \to \infty$, $\Phi(r) < r_1^{a_1} + \cdots + r_n^{a_n}$. It is obvious that for $a' \in B_\rho$ the set B_ρ contains the entire hyperoctant $\{a : a_i > a_i', \ i=1, \cdots, n\}$. On the other hand, if $a' \notin B_\rho$, then any point $a \in \boldsymbol{R}_+^n$ with $a_i \leqq a_i'$ is not in B_ρ." Since for any given j there exist vectors r with arbitrary large length $|r|$ such that $0 < r_j < 1$ the "obvious" part of this statement is not too obvious (but true) and the "On the other hand" part is wrong as the example $\Phi(r) = r_1 + \cdots + r_n$ shows. For the same reason the other limits of the same type should be explained more explicitly. Also the indeterminate factor 0^0 may occur in the formulas of Theorem

3.1.4 for arbitrarily large $|k|$. The same holds for similar formulas in the neighborhood of this theorem. On p. 197, formula (3.6.52) supposedly implies (3.6.53) "Since the function $|U(\lambda)|$ is subharmonic ". This is about as helpful as the advice "If in Chicago, turn left to find the loop". Hörmander's advice (p. 104) for the same conclusion is in no way better. In fact, the reader has to do a good deal of work to get from (3.6.52) to (3.6.53). On p. 213, the following statement is made: "the area of an analytic complex one-dimensional surface M is equal to the sum of the areas of its projections onto the coordinate planes". This is true for the area elements, but not for the areas themselves, and it does not become true because a good number of mathematicians have made this erroneous statement. A naive beginner would conclude that the area of any complex curve in the unit polydisc is bounded by πn.

Integration over analytic sets is badly handled. At first the integral is only defined for intersections of the analytic set with polydiscs and three pages later it is used for intersections with balls. The author may believe that his Lemma 4.1.1 proves that a continuous form ω of bidegree $(n-1, n-1)$ is locally integrable over an analytic set of dimension $n-1$.

Surely, the sheet number N on p. 221 is finite over each component of $\mathscr{E}'_r - \chi'''_{f,n}$; but there may be infinitely many components. Hence N may not be bounded. Therefore the sum in (4.1.22) may be pointwise finite, but may not have a bounded length. So the existence of the integral (4.1.22) is not established. More and different work is required to prove the integrability theorem for continuous forms on analytic sets. Integration over certain real analytic sets of dimension $n-1$ occurs in (4.3.23) and in Theorem 4.3.6. Integration over real analytic sets is complicated by the fact that the singular set may have codimension one, and that the regular set does not carry a natural orientation or may not even be orientable. Nothing at all is said about the integration occurring in (4.3.23). In [**5**, Lemma 10.4] an attempt was made to explain this integration. For the integration in Theorem 4.3.6 only Lemma 4.1.2 is given without proof. Here the author should have done better.

The translators did a fine job. The printing is narrow. Some subscripts of less than half a millimeter are used! The buyer is advised to acquire a magnifying glass as well. There are quite a number of misprints. The worst of these is the conversion of complicated exponents into factors on pp. 107, 108 in the English edition. For instance, the last formula on p. 107 reads: "Since, for any non-negative $a_1, \cdots, a_n$,

$$r_1 r_2 \cdots r_n = \prod_{i=1}^{n} (r_i^{a_i})^{1/a_i} \leqq (r_1^{a_1} + \cdots + r_n^{a_n}) \sum_{1}^{n} a_i^{-1},\text{''}$$

Obviously the formula is wrong, since $r_1 = \cdots = r_n = 1 < n$ and $a_i = n^3$ for all $i = 1, \cdots, n$ imply $1 \leqq 1/n$. At first all a_j have to be positive. Then $\xi = \sum_{i=1}^{n} 1/a_i$ is defined and belongs in the exponent (as is also true of the first formula on the next page). A correct formula is

$$r_1 r_2 \cdots r_n \leqq (r_1^{a_1}/a_1\xi + \cdots + r_n^{a_n}/a_n\xi)^{\xi}.$$

Despite these shortcomings, this book provides a good introduction to the subject matter. The beginner can learn much from it and the expert can use it as a reference book. It will have its impact on the future of the field.

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Elliptic modular functions, by B. Schoeneberg, Springer-Verlag, New York, Heidelberg and Berlin, 1974, 229+viii pp.

Before the appearance of Gunning's *Lectures on modular forms* in 1962—if one leaves aside Hardy's 1940 book, *Ramanujan*, which does not attempt to deal with the theory of modular functions systematically, but instead treats the subject with the characteristically unusual (though always interesting) perspective of the great Indian mathematician in mind—the only book available in the English language in this important area of mathematics was Lester Ford's classic, *Automorphic functions*. First published in 1929 as an elaboration of a 1915 Edinburgh Mathematical Tract, Ford's book served the mathematical public well for many years. It is hardly a criticism to point out the obvious—that by the early 1960's it was long out of date. While Ford deals quite effectively with uniformization theory and with the geometry of discontinuous groups—in particular he gives a lucid account of the construction of fundamental regions for discontinuous groups by what has come to be known as "Ford's method" of isometric circles—a number of fundamental developments in the decades following the publication of Ford's book created the need for a new exposition of the theory of modular and automorphic functions in one complex variable.

Though small in size and limited in intention, Gunning's book went far toward beginning to fill this need. Treating the modular group and certain congruence subgroups from the viewpoint of the theory of compact Riemann surfaces, Gunning made available to his readers an entire complex of ideas too "modern" to appear in Ford's work. Notable examples are the application of the Riemann-Roch theorem to calculate the dimension of the space of cusp forms, the introduction of the Petersson inner product and