15. I. N. Sneddon, *The special functions of mathematical physics and chemistry*, Oliver and Boyd, Edinburgh, 1961.

16. B. Spain and M. G. Smith, *Functions of mathematical physics*, Van Nostrand Rheinhold, London, 1970.

17. G. Szegö, *Orthogonal polynomials*, 3rd ed., Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, R.I., 1967.

18. J. D. Talman, *Special functions: A group theoretic approach*, Benjamin, New York and Amsterdam, 1968. MR **39** #511.

19. C. A. Truesdell, *An essay toward a unified theory of special functions based upon the functional equation* $(\partial/\partial z)F(z,\alpha) = F(z,\alpha+1)$, Ann. of Math. Studies, no. 18, Princeton Univ. Press, Princeton, N.J., 1948. MR **9**,431.

20. N. Ja. Vilenkin, *Special functions connected with class one representations of groups of motions in spaces of constant curvature*, Trudy Moskov. Mat. Obšč. **12** (1963), 185–257 Trans. Moscow Math. Soc. **1963**, 209–290. MR **29** #191.

21. ———, *Special functions and the theory of group representations*, "Nauka", Moscow, 1965; English transl., Transl. Math. Monographs, vol. 22, Amer. Math. Soc., Providence, R.I., 1968. MR **35** #420; **37** #5429.

22. G. N. Watson, *A treatise on the theory of Bessel functions*, 2nd ed., Cambridge Univ. Press, Cambridge; Macmillan, New York, 1944. MR **6**,64.

23. E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 4th ed., Cambridge Univ. Press, London, 1927.

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BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 2, March 1976

Essentials of Padé approximants, by George A. Baker, Jr., Academic Press, New York, 1975, xi+306 pp., $26.00.

The area of rational approximation and interpolation of functions has been studied intensively since the advent of electronic computers. This has brought the Padé table to the foreground and the text under review is the first pulling together of a lot of information about these tables that has appeared in the last 20 years. The texts by Perron and Wall on continued fractions, each of which devotes a chapter to the Padé table, have been among the chief references so far. A rational function $r_{m,n}(z)=p_{m,n}(z)/q_{m,n}(z)$ is of type (m, n) if $p_{m,n}(z)$ is a polynomial of degree $\leqq m$ and $q_{m,n}(z)$ a polynomial of degree $\leqq n$. $r_{m,n}(z)$ interpolates a given function $f(z)$ at the distinct points $z_1, \cdots, z_k$ if $r_{m,n}(z_i)=f(z_i)$, $i=1, \cdots, k$. If some of the points z_i coincide, say $z_1=z_2=z_3$, then it is natural to require $r_{m,n}(z_1)=f(z_1)$, $r'_{m,n}(z_1)=f'(z_1)$, and $r''_{m,n}(z_1)=f''(z_1)$ instead of $r_{m,n}(z_i)=f(z_i)$ for $i=1, 2, 3$. The case $z_1=z_2=\cdots=z_k$, i.e. $r^{(i)}_{m,n}(z_1)=f^{(i)}(z_1)$, for $i=0, 1, \ldots, k-1$ requires that $r_{m,n}(z)$ has a high order of contact with $f(z)$ at z_1. There are two classical and equivalent definitions of the (m, n) Padé approximant $R_{m,n}$ to $f(z)$ at $z=0$:

1. find the unique rational function $R_{m,n}$ in lowest terms such that $f(z)-R_{m,n}(z)=O(z^k)$, $k=$maximum, and
2. find polynomials $P_{m,n}$ and $Q_{m,n}$ such that $Q_{m,n}(z)f(z)-P_{m,n}(z)= O(z^{m+n+1})$, and let $R_{m,n}$ be $P_{m,n}/Q_{m,n}$ in lowest terms.

In definition 1, $R_{m,n}$ depends on $m+n+1$ parameters and one would expect max $k \geqq m+n+1$. That this need not be the case is seen by the example $f(z)=1+z^2$, $m=n=1$, $R_{1,1}\equiv 1$. In definition 2, $P_{m,n}$ and $Q_{m,n}$ may not be unique but the ratio $P_{m,n}/Q_{m,n}$ is. In either case $R_{m,n}$ exists and is unique. $O(z^k)$ means a power series starting with a term of degree $\geqq k$, $O(z^k)=d_k z^k+d_{k+1} z^{k+1}+\cdots$. It is customary to think of $f(z)$ as a power series: $f(z)=\sum_0^\infty c_n z^n$ with $c_0\neq 0$, and to normalize $R_{m,n}$ so that its denominator equals 1 at $z=0$. The Padé table of $\sum_0^\infty c_n z^n$ consists of all $R_{m,n}$ normalized as above. There is considerable variation in notation among writers on the Padé table as defined above. $R_{m,n}$ may signify a function whose numerator is of degree $\leqq m$ or $\leqq n$, it may or may not be in lowest terms, and the table may have columns labelled by m or by n, etc. The present text adopts a different definition and notation—referred to in this book as the modern definition. "We define the L, M Padé approximant to $A(x)$ by

$$[L/M]=P_L(x)/Q_M(x),$$

where $P_L(x)$ is a polynomial of degree at most L and $Q_M(x)$ is a polynomial of degree at most M. The formal power series $A(x)=\sum_{j=0}^\infty a_j x^j$ determines the coefficients of $P_L(x)$ and $Q_M(x)$ by the equation

$$A(x)-P_L(x)/Q_M(x)=O(x^{L+M+1}).$$

Since we can obviously multiply the numerator and denominator by a constant and leave $[L/M]$ unchanged, we impose the normalization condition $Q_M(0)=1.0$. Finally we require that P_L and Q_M have no common factor."

In a later chapter the existence of P_L/Q_M is discussed since $[L/M]$ may not exist for certain values of L and M. The equation $A(x)-P_L(x)/Q_M(x)=O(x^{L+M+1})$ leads to a set of linear equations with the a_j's as coefficients and with the coefficients of P_L and Q_M as unknowns. These equations are referred to throughout the book as the Padé equations. In turn, these equations lead to consideration of the persymmetric determinants

$$c(r/s)=\det\left| a_{r-1-s+i+j} \right|_{i,j=1,\cdots,s},$$

which are fundamental to the whole theory of Padé approximants.

The book is divided into four parts:

Part 1 on algebraic properties gives in its nine chapters (of 120 pp.) a large number of theorems on such topics as recurrence relations between Padé approximants; relation to continued fractions; matrix representation; identities for the determinants $c(r/s)$; formal expansions of hypergeometric functions; relationship to the classical orthogonal polynomials, etc.

Part 2 on convergence theory includes the classical results of de Montessus de Ballore as well as a large number of recent results by the author and others.

Part 3 on series of Stieltjes and Pólya includes 40 pp. on the standard material on Stieltjes series, convergence, moment problem, Padé table, integral representation, etc. These series are characterized by $f(x)=\sum_0^\infty a_j x^j$, where $a_j=(-1)^j\int_0^\infty u^j\, d\phi(u)$ and $\phi(u)$ is a bounded nondecreasing function with infinitely many points of increase. This representation of the a_j's enables us to give rather precise information about the determinants $c(r/s)$ and, therefore, about the Padé approximants, the associated continued fractions, upper and lower bounds and location of zeros and poles of the approximants.

The Pólya series are defined by

$$f(z)=a_0 e^{\gamma z}\prod_1^\infty(1+\alpha_j z)\Big/\prod_1^\infty(1-\beta_j z),$$

where $a_0>0$, $\gamma\geqq 0$, $\alpha_j\geqq 0$, $\beta_j\geqq 0$, $\sum^\infty(\alpha_j+\beta_j)<\infty$. They are treated in 8 pp. giving the basic results for these series.

Part 4 on generalizations and applications contains a series of short chapters concerning various open problems, generalizations to several dimensions, matrix series, and applications to important problems in physics.

Some comments on the presentation and style of writing are in order. The style is pleasantly casual but in places so much so that misunderstandings may arise. We give a number of examples that struck this reviewer. Nowhere is the symbol $O(x^k)$ defined, indicating a reasonably sophisticated reader, yet on p. 116 a literature reference is given for $\tan(\theta-\psi)=(\tan\theta-\tan\psi)/(1+\tan\theta\tan\psi)$. The reader is expected to know basic complex variables, yet on p. 169 is given an (incomplete) proof of the Bolzano-Weierstrass theorem. On p. 202 we find "$\bigcup_{n=1}^\infty E_n$ stand for union, i.e., every point in any E_n". On p. 194 the phrase "finite set" means a bounded, possibly infinite, set in the plane. On p. 128, $f(x)=(1+x)^{-1/2}$ seems to designate an analytic function, $x=$ complex, and also seems to refer to only one branch ($f(0)=+1$) since " $\cdots f(x)$ is not singlevalued and so may not be equal to its analytic continuation". P. 268, condition (c) of Theorem 20.1 reads "(c) $G(\rho_1)\geqq G(\rho_2)$, $\rho_1\geqq\rho_2$" and means "if $\rho_1\geqq\rho_2$ then $G(\rho_1)\geqq G(\rho_2)$" rather than "if $G(\rho_1)\geqq G(\rho_2)$ then $\rho_1\geqq\rho_2$." On pp. 195–196 in Lemma 14.2 and its proof, $f(z)$ refers to the function at hand, $f(t)$ refers to a different function (of equation (11.106)) and $|f(t)|$ is a fixed constant.

Some terminology, although imprecise, is very descriptive: P. 75, "The idea is to generate successively Padé approximants with successively higher denominators". P. 80 and elsewhere, "roots of a polynomial $c(z)$" instead of roots of an equation. Occasionally we find "zeros of polynomials". P. 91, "If we let Equation (7.20) keep track of the coefficients of $\cdots$". P. 115, "Reflection points (z, z'')" with respect to a circle. P. 176, "Hence it is necessary that $L+M+1$, the order of the first nonexact derivative, tends to infinity with k". P. 179, "Define $[t(x)]_n$ to be only those terms in x of degree higher than n." means $[t(x)]_n$ is the *sum* of those terms. P. 186, "The behavior of vertical or horizontal sequences of Padé approximants is quite well characterized, or characterizable, by rigorous theorems for functions whose closest nonpolar singularity is "smooth"."

While there seem to be few mistakes, we noted the following: P. 29, formulas (3.14) and (3.15) should be constants rather than proportional to x. On p. 137 in the proof of Lemma 11.2 we find "First observe that

$$(11.9)\qquad (-1)^n a_n=\left[\sum_{j=1}^{t}\sum_{k=1}^{m_j}A_{j,k}\binom{-k}{n}\right]y_j^n$$

is the binomial expansion of the function (in powers of z^{-1})

$$h(z)=\sum_{j=1}^{t}\left[\sum_{k=1}^{m_j}\frac{A_{jk}}{(z-y_j)^k}\right],"$$

It seems the numerator should be $A_{jk}z^k$ and that the proof must therefore be altered. The proof of the parabola theorem on p. 51 is not correct.

The book reflects the author's love and enthusiasm for the subject. It surely will be an important reference text in the field for years to come for physicists, engineers, chemists and mathematicians, pure and applied.

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BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 2, March 1976

Distributive lattices, by Raymond Balbes and Philip Dwinger, University of Missouri Press, Columbia, Missouri, 1975, xiii+294 pp., $25.00.

Lattice theory, as an independent branch of mathematics, has had a somewhat stormy existence during its hundred-odd years of being. Its origins are to be found in Boole's mid-nineteenth century work in classical logic; and the success of what we now call Boolean algebra in this field led to the late nineteenth century attempts at the formalization of all of mathematical reasoning, and eventually to mathematical logic.

Schröder and Peirce introduced the concept of an abstract lattice as a generalization of Boolean algebras, while Dedekind's work on algebraic numbers led him to the introduction of lattices outside of logic and to the concept of modular lattices. These late-nineteenth century investigations did not lead to widespread interest in lattice theory—it was not until the thirties that lattice theory truly became an object for independent and systematic study by mathematicians.

Stone's representation theory for Boolean algebras and distributive lattices, Menger's work on the subspace structure of geometries, von Neumann's coordinatization of continuous geometry and Birkhoff's recognition of the lattice as a basic tool in algebra were among the forces which combined in the late thirties to enable Birkhoff successfully to promote the idea that lattice theory is a branch of mathematics worthy of the attention of the community.

The very simplicity of the basic concepts in lattice theory and the degree of abstraction in its relationship to other branches of mathematics have proved to be at once both its strongest and weakest points.

Lattices are ubiquitous in mathematics. The beauty and simplicity of the abstraction and the ability to tie together seemingly unrelated pieces of mathematics are certainly appealing to the mathematician-as-artist. The introduction of new and nontrivial techniques for the solution of outstanding problems, for example in universal algebra, is mathematically rewarding; and the discovery of new questions which become natural to ask in the context of lattice theory is undoubtedly intriguing.

Through the vehicle of lattice theory one can hope to contribute to