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ON THE μ-INVARIANT OF Z-HOMOLOGY 3-SPHERES

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Consider, for each genus $n \geqslant 0$, an oriented handlebody U of genus n, embedded in the 3-sphere Σ in such a way that the closure U' of $\Sigma - U$ is homeomorphic to U. Let H be the group of all orientation-preserving homeomorphisms of Bd $U \longrightarrow$ Bd U, and let F (respectively L) be the subgroup of those maps in H which extend to homeomorphisms of U (respectively U'). Let $\widetilde{H}$ be the induced group of automorphisms of H_1(Bd U), and let K be the kernel of the natural map $\eta: H \longrightarrow \widetilde{H}$. Each element $h \in H$ may be used to define a 3-manifold $M(h)$, which is represented as the disjoint union of U and $-U$, identified by the rule $x = h(x)$ for each $x \in$ Bd U, i.e. a Heegard splitting $S(h)$ of genus n. Moreover, each 3-manifold admits such a representation for some n and some $h \in H$.

The μ-invariant $\mu(M(h))$ is defined in [5]. For known results see [2], [6], [7]. The purpose of this note is to announce results of a study of $\mu(M(h))$, which relate the value of $\mu(M(h))$ to the membership of h in various subgroups of H which are closely related to the groups F, L, K defined above. Our main results are:

I. We give a constructive procedure for enumerating all **Z**-homology spheres with μ-invariant ½. The enumeration proceeds by enumerating all pairs (W, g), where W is an $n \times n$ symmetric unimodular matrix which has even diagonal entries and signature 8 (mod 16) and g is an arbitrary element of $L \cap K$. Each such pair determines a map $h = h(W, g) \in H$ which determines a Heegaard splitting $S(h)$ of a 3-manifold $M(h)$ with $\mu(M(h)) = ½$. The Heegard genus n of $M(h)$ will always be at least 8.

II. We give a constructive procedure for obtaining, for each even Heegaard genus $n \geqslant 2$, infinitely many **Z**-homology spheres with μ-invariant ½, each being given as $S(h)$ for some $h \in H$.

III. THEOREM. *For each $n \geqslant 2$ and each $h \in H$ such that $H_1(M(h), \mathbf{Z}/2\mathbf{Z}) = 0$, there is a normal subgroup $P(h)$ of index* 2 *in K such that $\mu(M(kh)) = \mu(M(h))$ if and only if $k \in P(h)$.*

Here is a brief description of our methods. We consider equivalence classes of *map pairs*, where a map pair (h_2, h_3) is an element of $H \times H$, and map pairs

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(h_2, h_3), (h'_2, h'_3) are *equivalent* if there exist elements f_1, f_2, f_3 in $\mathcal{F}$ such that

$$h'_i = f_i h_i f_1, \qquad i = 2, 3. \tag{1}$$

With each such map pair (h_2, h_3) we associate a *fundamental triple* of oriented 3-manifolds (M_1, M_2, M_3), defined by the respective Heegaard splittings $S(h_3 h_2^{-1})$, $S(h_2)$, $S(h_3)$. Equivalent map pairs have fundamental triples which are topologically equivalent, and also they admit Heegaard splittings that are simultaneously equivalent in the particular manner dictated by equations (1). We also associate with each equivalence class $[(h_2, h_3)]$ of map pairs a *triadic 4-manifold* N which has the property that its oriented boundary is equivalent to the disjoint union $M_1 + M_2 - M_3$. The above constructions are all based upon ideas in [3] and [4]. In particular, the concept of (an equivalence class of) a map pair is abstracted from the concept of (an equivalence class of) a Heegaard presentation (see [3]) or a Heegaard representation (see [4]). It then follows from a theorem due to Rohlin [8] that

$$\mu(M_1) + \mu(M_2) - \mu(M_3) \equiv -\tau/16 \pmod 1, \tag{2}$$

where τ is the signature of the bilinear form $H_2(N)/\text{torsion} \times H_2(N)/\text{torsion} \longrightarrow \mathbf{Z}$, provided that a fundamental triple (M_1, M_2, M_3) is made up of $\mathbf{Z}/2\mathbf{Z}$-homology spheres and the bilinear form for N has even type.

An *abelianized map pair* is the image $(\eta(h_2), \eta(h_3))$ of a map pair (h_2, h_3) in $\widetilde{H} \times \widetilde{H}$. Equivalence of abelianized map pairs is defined in the obvious way. Using methods initiated in [1], we study abelianized map pairs. In particular, we show that $H_1(M_i; \mathbf{Z})$, $i = 1, 2, 3$, is completely determined by the equivalence class $[(\eta(h_2), \eta(h_3))]$, and also that if $H_1(M_3; \mathbf{Z}) = 0$, then τ can be read off from an appropriate canonical representative of $[(\eta(h_2), \eta(h_3))]$. Even more, we show that we can, to a certain extent, construct map pairs which have desirable properties. For example, for map pairs where the associated 4-manifold has a bilinear form of even type:

(i) We can arrange matters so that $\tau \equiv 8 \pmod{16}$, also $M_2 \approx M_3 \approx \Sigma$, also $H_1(M_1; \mathbf{Z}) = 0$. Since $\mu(\Sigma) = 0$, equation (2) then forces $\mu(M_1)$ to be ½.

(ii) We arrange matters so that M_2 and M_3 are given $\mathbf{Z}$-homology spheres with known μ-invariant, and also so that τ is either congruent to 0 (mod 16) or to 8 (mod 16), as desired, and finally so that $H_1(M_1; \mathbf{Z}) = 0$. Equation (2) then implies that if $\mu(M_2) - \mu(M_3)$ is incongruent to $\tau/16$ (mod 1), then $\mu(M_1) = ½$.

(iii) We arrange matters so that M_1 is a given $\mathbf{Z}/2\mathbf{Z}$-homology sphere, and so that M_2 and M_3 are each Σ. Then $\mu(M_1) \equiv -\tau/16 \pmod 1$ can be read off from the abelianized map pair.

Details and related matters will be discussed in another journal.

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