BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 2, March 1976

THREE WEIL REPRESENTATIONS ASSOCIATE TO FINITE FIELDS

BY PAUL GÉRARDIN[1]

Communicated by Robert M. Fossum, November 3, 1975

ABSTRACT. Following the method of A. Weil [4], we define the Weil representation of general linear groups in 1, of symplectic groups (odd characteristic) in 2, of unitary groups in 3, over finite fields. We give its character and decomposition and some functorial properties. The symplectic case was also studied by R. Howe [1] and M. Saito [3], the unitary case by R. I. Lehrer [2].

1. The Weil representations of symplectic groups (odd characteristic).

1.1. Let (E, j) be a symplectic vector space over the field k with q elements. Let $H(E, j)$ be the group $E \times k$ with the law

$$(w, z)(w', z') = (w + w', z + z' + i(w, w')), \tag{1}$$

where $i = j/2$. It is a two-step nilpotent group with center Z isomorphic to k by $z \longmapsto (0, z)$. The group $\mathrm{Sp}(E, j)$ of j acts on $H(E, j)$ by s: $(w, z) \longmapsto (sw, z)$.

1.2. For each nontrivial character ζ of Z, there is a unique class $\eta_\zeta^{(E,j)}$ of irreducible representations of $H(E, j)$ given by ζ on Z.

THEOREM 1. *There is a unique extension* $W_\zeta^{(E,j)}$ *of* $\eta_\zeta^{(E,j)}$ *to* $\mathrm{Sp}(E, j)$, *except for* $q = 3$, $\dim E = 2$, *where there is a unique extension* $W_\zeta^{(E,j)}$ *disjoint from its conjugate.*

The representation $W_\zeta^{(E,j)}$ is called the Weil representation of $\mathrm{Sp}(E, j)$ associated to the character ζ.

1.3. The Weil representations $W_\zeta^{(E,j)}$ have the following properties:

(1) $W_\zeta^{(E,j)} = W_{\zeta'}^{(E,j)}$ iff $\zeta'((0, z)) = \zeta((0, zt^2))$ for a $t \in k^*$ and all $z \in k$.

(2) $W_\zeta^{(E,j)}$ splits in two simple components of degree $(q^n + 1)/2$ and $(q^n - 1)/2$, where $n = \dim E$, given on the center of $\mathrm{Sp}(E, j)$ respectively by $(1/q)^n$ and $-(1/q)^n$.

(3) The complex conjugate of $W_\zeta^{(E,j)}$ is $W_{\zeta^{-1}}^{(E,j)}$.

(4) The support of the character of $W_\zeta^{(E,j)} \otimes \eta_\zeta^{(E,j)}$ is the set of conjugates of $\mathrm{Sp}(E, j)Z$.

(5) The class $W_\zeta^{(E,j)} \otimes W_{\zeta^{-1}}^{(E,j)}$ is the natural representation of $\mathrm{Sp}(E, j)$ in $\mathbf{C}[E]$.

AMS (MOS) subject classifications (1970). Primary 20C15, 20G40; Secondary 10C30.
[1]Supported in part by NSF Grant MPS72-05055A02.

(6) For $s \in \mathrm{Sp}(E, j)$, Tr $W_\zeta^{(E,j)}(s)$ has $q^{N(s)}$ for absolute value, $2N(s) = \dim \operatorname{Ker}(s-1)$.

(7) Let E_1 be an isotropic subspace of E and $E_0 = E_1^\perp/E_1$. There is a natural surjective map from the stabilizer L of E_1 in $\mathrm{Sp}(E, j)H(E, j)$ onto $\mathrm{Sp}(E_0, j_0)H(E_0, j_0)$, where j_0 is the symplectic form induced by j on E_0. Let P be the stabilizer of E_1 in $\mathrm{Sp}(E, j)$. The representation π of $PH(E, j)$ induced by the representation $W_\zeta^{(E_0,j_0)} \otimes \eta_\zeta^{(E_0,j_0)}$ of L is the product of the restriction of $W_\zeta^{(E,j)} \otimes \eta_\zeta^{(E,j)}$ to $PH(E, j)$ by the character $\chi^{E_1}(s) = (\det_{E_1} s)^{(1-q)/2}$. Moreover $W_\zeta^{(E,j)}$ is the only extension of $\eta_\zeta^{(E,j)}$ to $\mathrm{Sp}(E, j)$ which is given by $\chi^{E_1} \otimes \pi$ on P.

(8) If $E = \Sigma E_r$ (orthogonal sum), and j_r is the restriction of j to E_r, the restriction of $W_\zeta^{(E,j)}$ to the product $\Pi\, \mathrm{Sp}(E_r, j_r)$ is $\bigotimes W_\zeta^{(E_r,j_r)}$.

(9) If $E = \mathrm{Res}_{k'/k} E'$ and j' is a symplectic form on E' such that $j = \mathrm{Tr}_{k'/k} j'$, the restriction of $W_\zeta^{(E,j)}$ to $\mathrm{Sp}(E', j')$ is $W_{\zeta'}^{(E',j')}$ where $\zeta'((0, z')) = \zeta((0, \mathrm{Tr}_{k'/k} z'))$.

(10) If $t \in \mathrm{Sp}(E, j)$ is semisimple, t belongs to a subgroup isomorphic to a product of $k_r(\pm)$, where $k_r(-)$ is the multiplicative group of the extension of degree r of k, and $k_r(+)$ is the kernel of the norm from k_{2r}^* to k_r^*. The trace of $W_\zeta^{(E,j)}$ on t is $\chi(t)(-1)^{a(t)}q^{N(t)}$, with $\chi(t) = \Pi\, t_r^{(1\pm q^r)/2}$ where t_r is the component of t in $k_r(\pm)$, $a(t)$ is the number of r such that $t_r \neq 1$. The other elements of $\mathrm{Sp}(E, j)$ are in proper subgroups of type P as in (7), and the character of $W^{(E,j)}$ on them is obtained from the formula of induced characters.

2. The Weil representations of unitary groups.

2.1. Let K/k be a quadratic extension of the field k with q elements, F a vector space over K and i a nondegenerate skew-hermitian form on F. The set of all couples (w, z), $w \in F$, $z \in K$ with $z - \bar{z} = i(w, w)$, is a group by the law (1); it is a two-step nilpotent group $H(F, i)$ with center Z isomorphic to k by $z \longmapsto (0, z)$. The group $U(F, i)$ of the form i acts on $H(F, i)$ by u: $(w, z) \longmapsto (uw, z)$.

2.2. For each nontrivial character ζ of the center Z of $H(F, i)$ there is a unique class $\eta_\zeta^{(F,i)}$ of irreducible representations of $H(F, i)$ given by ζ on Z.

THEOREM 2. *There is a unique extension $W^{(F,i)}$ of $\eta_\zeta^{(F,i)}$ to $U(F, i)$ such that*:

(a) *for q odd, $W^{(F,i)} = \chi^{(F,i)} \otimes W_\zeta^{(E,j)}$ on $U(F, i)$, where $\chi^{(F,i)}(u) = (\det u)^{(1+q)/2}$, E is the underlying k-vector space of F, $j = \mathrm{Tr}_{K/k}\, i$, and $W_\zeta^{(E,j)}$ is the Weil representation of* $\mathrm{Sp}(E, j)$ *associated to ζ*;

(b) *for q even, $W^{(F,i)}$ is real, and moreover for $q = 2$, $\dim F = 2$, $W^{(F,i)}$ contains no one dimensional representation of $U^{(F,i)}$ which factors through the determinant. The representation $W^{(F,i)}$ is called the Weil representation of $U(F, i)$.*

2.3. The Weil representation $W^{(F,i)}$ has the following properties:

(1) $W^{(F,i)}$ does not depend on ζ.

(2) Let $n = \dim_K F$; $W^{(F,i)}$ splits in q simple classes of degree $[q^n - (-1)^n]/(q+1)$ corresponding to the nontrivial characters of the center of $U^{(F,i)}$ and, for $n > 1$, a simple class of degree $q[q^{n-1} - (-1)^{n-1}]/(q+1)$.

(3) For $u \in U^{(F,i)}$, Tr $W^{(F,i)}(u) = (-1)^n(-q)^{N(u)}$, where $N(u) = \dim_K \operatorname{Ker}(u-1)$.

(4)–(10) As in part 1, (4)–(10) with the obvious modifications and all the characters χ are now trivial; in (9) $F' = \operatorname{Res}_{K'/K} F$ and $[K' : K]$ is odd, $i = \operatorname{Tr}_{K'/K} i'$.

3. **The Weil representations of general linear groups.** If the class W^V of the natural representation of $\mathrm{GL}(V)$, for a finite dimensional vector space over the field k of order q, in the space of complex functions on V is called the Weil representation of $\mathrm{GL}(V)$, its properties are similar to those of the Weil representation of the unitary group of same rank over k (for example, (2) and (4) up to the sign $(-1)^n$).

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SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540

UER DE MATHÉMATIQUES, UNIVERSITÉ PARIS VII, 2 PL. JUSSIEU, 75005 PARIS, FRANCE