

PARTIAL AND COMPLETE CYCLIC ORDERS

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We show that, in contrast to a famous theorem on linear orders, not every partial cyclic order on $M = \{1, \dots, m\}$ can be extended to a complete cyclic order. In fact, the complexity, in a certain sense, of sufficient conditions for such an extendability increases rapidly with m .

DEFINITION 1. (i) Two linear orders, (a_1, \dots, a_m) and (b_1, \dots, b_m) , on M are called *cyclically equivalent* if there exists $k \in M$ such that $[j - 1 \equiv (i - 1 + k) \pmod{m}] \Rightarrow a_i = b_j$.

(ii) A *complete cyclic order* (CCO) on M is an equivalence class C of linear orders modulo cyclic equivalence; denote $a_1 a_2 \cdots a_m$ for the equivalence class containing (a_1, a_2, \dots, a_m) .

DEFINITION 2. A *partial cyclic order* (PCO) on M is a set Δ of cyclically ordered triples (COTs) out of M such that:

- (i) $xyz \in \Delta \Rightarrow zyx \notin \Delta$ ("antisymmetry"),
- (ii) $\{xyz, xzw\} \subset \Delta \Rightarrow xyw \in \Delta$ ("transitivity");

since $xyz = zxy$, etc., also $yzw \in \Delta$ is implied.

THEOREM 3. (i) If C is a CCO then the set Δ of all COTs derived from C is a PCO. (ii) If Δ is a saturated PCO, i.e., $\{x, y, z\} \in \binom{M}{3}$ & $xyz \notin \Delta \Rightarrow zyx \in \Delta$, then there exists a CCO from which all of Δ 's COTs are derived; Δ is then said to be extendable to a CCO.

COROLLARY 4. A PCO is extendable to a CCO if and only if it is contained in a saturated PCO.

It is natural to ask whether every PCO is extendable to a CCO (or, equivalently, is contained in a saturated PCO). In view of the following example, the answer is in the negative.

EXAMPLE 5. Let $M = \{a, b, \dots, m\}$ be the set of the first thirteen letters, and let $\Delta = \{acd, bde, cef, dfg, egh, fha, gac, hcb, abi, cij, bjk, ikl, jlm, kma, lab, mbc, hcm, bhm\}$. Obviously, Δ is a PCO. Suppose that $\Delta^* \supset \Delta$ is a saturated PCO. If $abc \in \Delta^*$ then, since $acd \in \Delta^*$, also $bcd \in \Delta^*$. Then, also $cde \in \Delta^*$, and successive applications of transitivity finally yield $acb \in \Delta^*$, which contra-

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dicts antisymmetry. Thus, $abc \notin \Delta^*$ and, therefore, by saturatedness, $acb \in \Delta^*$. Analogously, since $abi \in \Delta^*$, also $cbi \in \Delta^*$, and successive applications of transitivity finally yield $abc \in \Delta^*$. Thus, antisymmetry is contradicted again. It follows that there is no saturated PCO that contains Δ .

The unextendability of Δ in Example 5 followed essentially from the fact that neither abc nor cba belongs to any PCO that contains Δ . That gives rise to the following definition.

DEFINITION 6. If $\tau = \{i, j, k\}$, $1 \leq i < j < k \leq m$, denote $\tau^+ = ijk$ and $\tau^- = kji$ for the two possible cyclic orderings of τ . A PCO Δ is said to satisfy the n th order condition if for every $\tau_1, \dots, \tau_n \in \binom{M}{3}$ there exists a PCO $\Delta^* \supset \Delta$ and $\epsilon_i \in \{+, -\}$ ($i = 1, \dots, n$), such that $\{\tau_1^{\epsilon_1}, \dots, \tau_n^{\epsilon_n}\} \subset \Delta^*$.

Obviously, all the n th order conditions ($n = 0, 1, \dots$) are necessary for extendability to a CCO and, as n increases, the n th order condition becomes stronger. The conjunction of all the n th order conditions ($n = 0, 1, \dots$) is a sufficient condition for every m . It is natural to ask whether there exists an n such that the n th order condition suffices for every PCO Δ on a finite set M to be extendable at a CCO. Unfortunately, the answer to this question also is in the negative. A sequence of PCOs that prove this is constructed as follows.

EXAMPLE 7. Let $m_0 = 13$ and let Δ_0 be the PCO on $M_0 = \{1, \dots, 13\}$ defined in Example 5 (identify a with 1, b with 2, etc.). As we have already seen, Δ_0 is not extendable to a CCO. However, since it is a PCO, it satisfies the 0th order condition. For the purpose of later use in induction, note that $\Delta_0 \setminus \{egh\}$ is extendable to the following complete cyclic ordering: $afbhcgdeijklm$. Suppose, by induction, that Δ_n is a PCO on $M_n = \{1, \dots, m_n\}$, Δ_n satisfies the n th order condition but is not extendable to a CCO. Suppose also that $xyz \in \Delta_n$ is such that $\Delta_n \setminus \{xyz\}$ is extendable to a CCO. We construct Δ_{n+1} as follows. Define $m_{n+1} = m_n + 15$ and $M_{n+1} = \{1, \dots, m_{n+1}\}$ and denote $(u_1, \dots, u_5, v_1, \dots, v_5, w_1, \dots, w_5) = (m_n + 1, \dots, m_{n+1})$. Let

$$\begin{aligned} \Delta' = (\Delta_n \setminus \{xyz\}) \cup \{ & zu_1u_2, yu_2u_3, u_1u_3u_4, u_2u_4u_5, u_3u_5x, u_4xy, xyv_1, \\ & u_1v_1v_2, yv_2v_3, v_1v_3v_4, v_2v_4v_5, v_3v_5x, v_4xy, v_5yu_1, \\ & u_5yw_1, zw_1w_2, yw_2w_3, w_1w_3w_4, w_2w_4w_5, w_3w_5u_5, \\ & w_4u_5y, w_5yz\}. \end{aligned}$$

Let Δ_{n+1} be the transitive closure of Δ' , i.e., Δ_{n+1} is the intersection of all the transitive classes of COTs that contain Δ' (see Definition 2). It turns out that Δ_{n+1} is a PCO, but is not extendable to a CCO. Also, $\Delta_{n+1} \setminus \{w_3w_5u_5\}$ is extendable to a CCO. The proof of these facts follows from the analogous properties of Δ_n . The important property of Δ_{n+1} is that it satisfies the $(n + 1)$ st order condition. A detailed proof will be given elsewhere. Here, we indicate that two cases are distinguished when a set of 3-element subsets of M_{n+1} is given.

First, when $|\tau_i \cap M_{n+1}| \geq 2$ for $i = 1, \dots, n+1$, the $\epsilon_i - s$ are determined essentially by the CCO on M_n to which $\Delta_n \setminus \{xyz\}$ is extendible. Otherwise, the induction hypothesis is applied and the $\epsilon_i - s$ are determined essentially by a PCO Δ^* that contains Δ_n and n of the $\tau_i - s$.

In view of Example 7, an algorithm for extending a PCO to a CCO which is based on successive additions of COTs, cannot be polynomial. We conjecture that there is no polynomial algorithm for this problem; note that there seems to be an equivalence between our problem and that of the Hamiltonian path, from the point of view of complexity of computations.

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