COBORDISM OPERATIONS AND SINGULARITIES OF MAPS
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If $f$ is a differentiable map of smooth manifolds, the critical set $\Sigma(f)$ is not a manifold, in general. However, there is a canonical resolution of the singularities of $\Sigma(f)$ (for generic $f$), due to I. Porteous [6]. This resolution can be used to give a geometric description of T. tom Dieck's Steenrod operations in unoriented cobordism [7]. This was suggested to me by Jack Morava, as a parallel to my discription of ordinary mod 2 Steenrod operations using branching cycles of maps of $n$-circuits [5].

1. Singularities of vector bundle maps. Let $\xi^n = (E \to X)$ and $\eta^p = (F \to X)$ be real vector bundles over the smooth manifold $X$ (without boundary), and let $g: E \to F$ be a vector bundle map. That is, $g$ is smooth, and for each $x \in X$, $g$ sends the fiber $E_x$ to the fiber $F_x$ by a linear map $g_x$. The critical set $\Sigma(g)$ is $\{x \in X, \text{rank}(g_x) < \min(n, p)\}$. Let $P(\xi) = (P(E) \to X)$ be the projectification of $\xi$, i.e. the bundle whose fiber over $x$ is the set of one-dimensional subspaces of $E_x$. Set $\tilde{\Sigma}(g) = \{l \in P(E), l \subset \text{kernel}(g)\}$. The projection $\tilde{\Sigma}(g) \to X$ is proper, and if $n < p$, its image is $\Sigma(g)$. (If $n > p$, its image is all of $X$.)

Lemma. (a) If $g: \xi^n \to \eta^p$ is a generic vector bundle map [4] over the $d$-manifold $X$, $\tilde{\Sigma}(g)$ is a $(d - i)$-manifold, where $i = p - n + 1$.

(b) If $h: \xi^n \to \eta^p$ is another such map, $\tilde{\Sigma}(h) \to X$ is properly cobordant with $\tilde{\Sigma}(g) \to X$.

This lemma is proved by considering the canonical bundle map $G$ over $\text{Hom}(\xi, \eta)$. A vector bundle map $g: \xi \to \eta$ defines a section of $\text{Hom}(\xi, \eta) \to X$, and $\tilde{\Sigma}(g) \to X$ is the pull-back of $\tilde{\Sigma}(G) \to \text{Hom}(\xi, \eta)$ by this section.

It follows from Quillen's geometric description of smooth unoriented cobordism theory $N^*$ [3] that this construction defines a natural transformation $\sigma: K(X) \to N^*(X)$. If $K(X)$ is defined as the set of all pairs $(\xi, \eta)$ of bundles over $X$, modulo the relation $((\xi \oplus \xi, \eta \oplus \xi) \sim (\xi, \eta))$, $\sigma$ is induced by $(\xi, \eta) \mapsto \tilde{\Sigma}(g)$, where $g: \xi \to \eta$ is a generic map. A dual $\overline{\sigma}$ is defined by $\overline{\sigma}[\xi, \eta] = \sigma[\eta, \xi]$.

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σ determines a family of stable cobordism characteristic classes σ_i, i ∈ Z, by setting σ_i(η^n) = σ(e^n, η^n) ∈ N^i(X), where e^n is the trivial n-bundle over X, n = p - i + 1. If \( \xi \) is a stable inverse for \( \xi \), \( \sigma_i(\xi) = \overline{\sigma}_i(\xi) \).

2. **Steenrod operations.** Thom's definition of characteristic classes gives a bijection between stable operations on \( N^* \) and stable \( N^* \) characteristic classes. Let \( \theta^i \) be the operation corresponding to the characteristic class \( \sigma_i \).

Our main result is the following relation between \( \theta = \Sigma \theta^i \) and tom Dieck's internal Steenrod operation \( R \) [7, p. 394]. Let \( P^{i-1} \) be the cobordism operation of degree \(-i\) which sends \( Z \rightarrow X \) to the composition \( (R P^i \times Z) \rightarrow Z \rightarrow X \), where \( R P^i \) is real projective \( i \)-space.

**Theorem \( \theta \) = PR.**

In other words, if \( \alpha \in N^q(X) \), \( \theta^i(\alpha) = \Sigma_i P^{i-1} R^i(\alpha) \). Since \( P^{i-1} = 0 \) for \( j < i \) and \( R^i(\alpha) = 0 \) for \( j > q \), this sum is finite.

It follows that \( \theta \) corresponds to the "expanded square" operation in unoriented piecewise-linear cobordism [1].

This theorem is a consequence of the observation that \( \overline{\sigma}_i(\xi^n) = \pi_n(e^n + i - 1) \) for \( i > -n \), where \( \pi: P(E) \rightarrow X \) is the projection and \( e \) is the cobordism Euler class of the (dual) canonical line bundle on \( P(E) \). (For \( i \leq -n \), \( \overline{\sigma}_i(\xi) \) is represented by \( P(\xi \oplus e^k) \), \( k = -n - i + 1 \).)

**Remark.** Conner and Floyd's cobordism Stiefel-Whitney classes \( w_i(\xi) \) (cf. [3]) are defined by the relation \( \Sigma_i (\pi^*w_i)e^{n-i} = 0 \). Thus \( \Sigma_i w_i \overline{\sigma}_{k-i} = 0 \) for \( k > 0 \).

3. **Bordism operations (cf. [5]).** There are dual actions of both \( \theta \) and \( R \) on smooth unoriented bordism theory \( N_* \). If \( M \) is a closed \( n \)-manifold, and \( [M] \in N_n(M) \) is the class of the identity map, \( \theta^i[M] \) is represented by \( \Sigma(df) \), where \( f: M^n \rightarrow R^{n+i-1} \) is a generic smooth map. The following result is analogous to Thom's nonembedding theorem using ordinary Steenrod operations.

**Corollary 1.** If the locally triangulable space \( X \) immerses topologically in \( R^n \), then \( R^i \) is zero on \( N_j(X) \) for \( i + j > n \).

The action of \( R^i \) on the bordism of a point is given by the "quadratic construction"

\[ Q_k(M) = M \times M \times S^{k-1}/(x, y, s) \sim (y, x, -s), \quad k = -n - i + 1. \]

**Corollary 2.** If \( M \) is a closed manifold, \( Q_k(M) \) is cobordant with \( P(TM \oplus e^k) \), \( k \geq 1 \).

In fact, \( M \times M \times D^k/(x, y, s) \sim (y, x, -s) \) minus an open tubular neighborhood of \( \{[x, x, 0]\} \) is a cobordism between them. This generalizes an argument of Conner and Floyd for \( k = 1 \) [2, p. 62].
REMARK. “Steenrod” operations in complex cobordism can be defined in the same way as $\theta^i$, by using complex vector bundles. Furthermore, replacing lines in $\xi$ by $k$-planes in $\xi$ yields a family of geometric operations $\theta^i_{(k)}$ for each $k$.

REFERENCES


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