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EXISTENCE THEOREMS FOR PARETO OPTIMIZATION IN BANACH SPACES

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1. **Introduction.** Pareto-type optimization problems with vector valued cost criteria are of importance in Economics (Pareto, 1896). Recently, there has been an extensive study of necessary conditions for optimality in such problems. We only mention here Smale [3], Weinberger [4], and Yu and Leitmann [5]. In the present paper we state existence theorems for Pareto optima in the very general setting of Banach spaces and ordering defined by cones.

2. **The main statement.** Let Z be a Banach space and Λ be a closed convex cone in Z. Given $x, y \in Z$, we write $x <_\Lambda y$ if $y - x \in \Lambda$. Given a nonempty set A of Z, an element $x_0 \in Z$ is said to be a weak Λ-extremum of A if x_0 belongs to the weak closure of A and there is no $x \in A$, $x \neq x_0$, with $x <_\Lambda x_0$. If Z^* is the dual of Z and Λ^* is the polar of Λ, that is, $\Lambda^* = [z^* \in Z^* | z^*\lambda \leqslant 0 \text{ for all } \lambda \in \Lambda]$, then x_0 is a Λ-extremum if and only if x_0 is in the weak closure of A, and for each $x \in A$, $x \neq x_0$, there is some element $\lambda^* \in \Lambda^*$ such that $\lambda^* x < \lambda^* x_0$. We shall assume below that Λ has the following angle property (ϵ): there is an element $a \in -\Lambda^*$ and a number ϵ, $0 < \epsilon \leqslant 1$, such that $\Lambda \subset [z | az \geqslant \epsilon \|a\| \, \|z\|]$.

Let (X, τ) be a topological space, $(B, \|\cdot\|)$ be a Banach space, and (G, ρ), (Y, d), (U, d') be metric spaces. We assume that (G, α, μ) is also a finite complete measure space with measure μ and σ-algebra α of ν-measurable sets. We also assume that Y and U are σ-compact, that is, countable union of compact subsets. Let A be any subset of $G \times Y$ such that for any $t \in G$ the set $A(t) = [y \in Y | (t, y) \in A]$ is nonempty. For each $(t, y) \in A$, let $U(t, y)$ be a nonempty subset of U. Let $f(t, y, u)$, $g(t, y, u)$ be given functions defined on the set $S = [(t, y, u) \in G \times Y \times U | (t, y) \in A, u \in U(t, y)]$ with values in B and Z respectively, or $f: S \longrightarrow B$, $g: S \longrightarrow Z$. We assume that A, S, f and g satisfy a Carathéodory condition (C) on G, that is, given $\epsilon > 0$ there is a compact subset $K \subset G$ with $\mu(G - K) < \epsilon$ such that the sets $A_K = [(t, y) \in A | t \in K]$, $S_K = [(t, y, u) \in S | t \in K]$ are closed and $f: S \longrightarrow B$, $g: S \longrightarrow Z$, restricted to S_K, are continuous (in the weak topology of B and Z respectively).

Given any Hausdorff space H, we denote by $\mu(G, H)$ the set of all μ-measurable functions on G with values in H. For any Banach space B, we denote by

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$L_1(G, B)$ the set of all strongly (Bochner) integrable μ-measurable functions z on G with values in B. Let L and M be two operators (not necessarily linear) defined on a nonempty subset X_0 of X, and having values in $L_1(G, B)$, $\mu(G, Y)$ respectively, for which we assume that, if $x_k \in X_0$, $x \in X$, and $x_k \longrightarrow x$ in X, then we have also $x \in X_0$, and there is a subsequence $[k_s]$ such that $Lx_{k_s} \longrightarrow Lx$ weakly in $L_1(G, B)$, and $Mx_{k_s} \longrightarrow Mx$ in measure in G as $s \longrightarrow \infty$.

For $(t, y) \in A$ we consider the sets

$$\tilde{\tilde{Q}}(t, y) = [(z^0, z) \in Z \times B | z^0 \in g(t, y, u) + \Lambda,$$

$$z = f(t, y, u), u \in U(t, y)], \qquad (t, y) \in A.$$

We say that the sets $\tilde{\tilde{Q}}(t_0, y), y \in A(t_0)$, satisfy property (Q) at (t_0, y_0) with respect to y, provided

$$\tilde{\tilde{Q}}(t_0, y_0) = \bigcap_{\epsilon > 0} \operatorname{cl} \operatorname{co} \bigcup [\tilde{\tilde{Q}}(t_0, y) | y \in A(t_0), d(y, y_0) \leqslant \epsilon].$$

We say that a pair $(x, u), x \in X_0, u \in \mu(G, U)$, is admissible provided $(Mx)(t) \in A(t)$, $u(t) \in U(t, Mx(t))$, $(Lx)(t) = f(t, Mx(t), u(t))$ μ-a.e. in G, and $g(\cdot, Mx(\cdot), u(\cdot)) \in L_1(G, Z)$. We consider the Z-valued functional $I[x, u] = \int_G g(t, Mx(t), u(t))\, d\mu$ defined on a class Ω of admissible pairs. We denote by $\{x\}_\Omega$ the set $[x \in X_0 | (x, u) \in \Omega$ for some $u]$. We say that the class Ω is closed (with respect to the Pareto-Lagrange problem under consideration) provided, whenever $(x_k, u_k) \in \Omega$, $k = 1, 2, \ldots, x_k \longrightarrow x$ in (X, Z) as $k \longrightarrow \infty$, $x \in X_0$, and there is some $u \in \mu(G, U)$ such that (x, u) is admissible, then there is also some $\bar{u} \in \mu(G, U)$ with $(x, \bar{u}) \in \Omega$ and $I[x, \bar{u}] <_\Lambda I[x, u]$. This definition is justified by lower closure theorems (cf. [1]). The class of all admissible pairs is closed. We assume that Z is reflexive.

THEOREM 1. *Under the assumptions above, let Ω be a nonempty closed class of admissible pairs such that $\{x\}_\Omega$ is relatively sequentially compact in X. Let there be a Z-valued function $\varphi(t)$, $t \in G$, $\varphi \in L_1(G, Z)$, such that $\varphi(t) <_\Lambda g(t, y, u)$ for all $(t, y, u) \in S$. Finally, let the sets $\tilde{\tilde{Q}}(t, y), y \in A(t)$, satisfy property* (Q) *with respect to y in $A(t)$, for almost all $t \in G$. Then, the Z-valued functional $I[x, u]$ has at least one Λ-Pareto optimum in Ω.*

Other existence theorems are given in [1]. In particular, variants of Theorem 1 are given in [2] which make it unnecessary to verify property (Q). Here we present a special case of Theorem 1 for Pareto-Lagrange problems in Euclidean spaces for ordinary differential equations.

3. **A special case.** For each t, $0 \leqslant t \leqslant 1$, let $A(t) \subset E^n$ and $U(t) \subset E^m$, be given closed subsets. Let $S = [(t, x, u) | 0 \leqslant t \leqslant 1, x \in A(t), u \in U(t)]$ be closed. Let B_0 be a compact subset of $A(0)$, and B_1 a closed subset of $A(1)$. Let $g(t, x, u) = (g_1, \ldots, g_N)$ and $f(t, x, u) = (f_1, \ldots, f_n)$ be given functions

continuous on S. Let Ω denote the set of all pairs of functions $x(t) = (x_1, \ldots, x_n)$, $u(t) = (u_1, \ldots, u_m)$, $0 \leqslant t \leqslant 1$, $x(\cdot)$ absolutely continuous, $u(\cdot)$ measurable on $[0, 1]$, satisfying the differential system $dx/dt = f(t, x(t), u(t))$ and the constraints $x(t) \in A(t)$, $u(t) \in U(t)$ a.e. in $[0, 1]$, $g(\cdot, x(\cdot), u(\cdot)) \in L_1[0, 1]$, and the boundary data $x(0) \in B_0$, $x(1) \in B_1$. We consider the problem of Pareto optimization in Ω (with respect to the positive cone $\Lambda = [(\lambda_1, \ldots, \lambda_N) | \lambda_i \geqslant 0, i = 1, \ldots, N]$ in E^N) for the vector valued cost functional $I[x, u] = (I_1, \ldots, I_N)$ with $I_j = \int_0^1 g_j(t, x(t), u(t))\, dt$, $j = 1, \ldots, N$. We need consider here the sets

$$\widetilde{\widetilde{Q}}(t, x) = [(z^1, \ldots, z^N, z^1, \ldots, z^n) | z^j \geqslant g_j(t, x, u), j = 1, \ldots, N,$$
$$z^i = f_i(t, x, u), i = 1, \ldots, n, u \in U(t)],$$

for $x \in A(t)$, $0 \leqslant t \leqslant 1$.

COROLLARY. *With the above notations, let the sets* $\widetilde{\widetilde{Q}}(t, \quad)$ *satisfy property* (Q) *with respect to* x *in* $A(t)$, *for almost all* $t \in [0, 1]$. *Let there be a continuous scalar function* $\Phi(\zeta)$, $0 \leqslant \zeta < +\infty$, with $\Phi(\zeta)/\zeta \longrightarrow +\infty$ as $\zeta \longrightarrow +\infty$, *and an index* j_0 *with* $1 \leqslant j_0 \leqslant N$ *such that* $g_{j_0}(t, x, u) \geqslant \Phi(|f(t, x, u)|)$ *for all* $(t, x, u) \in S$. *Further, let there be scalar functions* $\varphi_j \in L_1[0, 1]$ *such that* $g_j(t, x, u) \geqslant \varphi_j(t)$ *for all* $(t, x, u) \in S$, $j = 1, \ldots, N$, $j \neq j_0$. *Then, there is at least one* Λ-*Pareto optimal solution for* I *in* Ω.

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