

STABLE FACES OF A POLYTOPE

BY GOTTFRIED T. RÜTTIMANN¹

Communicated by Robert Bartle, December 1, 1975

The purpose of this announcement is to provide several new concepts for the study of the face-lattice of a polytope. The proofs of the results stated herein will appear elsewhere.

1. **Preliminaries.** Let P be a polytope with $\dim P \geq 0$ [4].

DEFINITION 1. A hyperplane S in $\text{aff } P$ is said to be a *support* if P is entirely contained in one of the closed halfspaces determined by S . $\{F(P), \leq\}$ denotes the poset of faces of P which is indeed a lattice (\wedge : infimum, \vee : supremum; 1 : greatest element; 0 : least element). $A \subseteq F(P)$ is called an *order ideal* if $a \leq b$ and $b \in A$ implies $a \in A$. The ideal $[0, a]$ ($= \{b \in F(P) \mid 0 \leq b \leq a\}$) is called a *principal order ideal (generated by a)*. A proper face a is called *split [1]* if $\{b \mid b \wedge a = 0\}$ is a principal order ideal and P is the direct convex sum of a and the generator of the principal order ideal. The faces 0 and 1 are called *split*.

A face of P is split if and only if it is a central element of the face-lattice.

DEFINITION 2. A pair of faces $a, b \in F(P) - \{1\}$ is said to be *orthogonal*, in symbols $a \perp b$, provided there exist two supports S_1, S_2 such that $a \subseteq S_1, b \subseteq S_2$ and $S_1 \cap S_2 = \emptyset$. We put: $a \perp 1$ ($1 \perp a$) if and only if $a = 0$. If $A \subseteq F(P)$, define $A^\perp = \{b \in F(P) \mid b \perp a \text{ for all } a \in A\}$. If $\{a\}^\perp$ is a principal order ideal, then a^\perp denotes its generator.

We have immediately: (i) $a \perp b \Rightarrow b \perp a$, (ii) $0 \perp a$ for all $a \in F(P)$, (iii) $a \perp a \Leftrightarrow a = 0$, (iv) $a \perp b \Rightarrow a \wedge b = 0$, (v) $a \leq b$ and $b \perp c \Rightarrow a \perp c$.

2. **Mutually stabilizing pairs of faces.**

DEFINITION 3. A face $a \in F(P)$ is said to be *stable* provided there exists a face b such that a is a maximal element in the subposet $\{\{b\}^\perp, \leq\}$. $S(P)$ denotes the set of stable faces of P . A pair $a, b \in F(P)$ is called *mutually stabilizing* in symbols $a \perp\!\!\!\perp b$, if a is a maximal element in $\{\{b\}^\perp, \leq\}$ and b is a maximal element in $\{\{a\}^\perp, \leq\}$.

Note that facets and split faces are stable. One easily gets: (i) $a \perp\!\!\!\perp b \Rightarrow$

AMS (MOS) subject classifications (1970). Primary 52A25, 06A10; Secondary 81A12.
 Key words and phrases. Stable face, split face, strong polytope, simplex, orthomodular poset, Boolean lattice, pure state, dispersion-free state, superposition of states.

¹Research supported by the Swiss National Science Foundation

$b \perp\!\!\!\perp a$, (ii) $a \perp\!\!\!\perp b \Rightarrow a \perp b$, (iii) $a \perp\!\!\!\perp b \Rightarrow a, b \in S(P)$, (iv) $0 \perp\!\!\!\perp a \iff a = 1$, $1 \perp\!\!\!\perp a \iff a = 0$.

THEOREM 1. *For every stable face a of a polytope there exists at least one face b such that the pair a, b is mutually stabilizing.*

THEOREM 2. *If $a \perp\!\!\!\perp b$ ($a, b \neq 1$) then there exists exactly one pair of supports S_1, S_2 such that $a \subseteq S_1, b \subseteq S_2$ and $S_1 \cap S_2 = \emptyset$.*

THEOREM 3. *$a \perp\!\!\!\perp b$ if and only if $a \perp b$ and $\dim(a \cup b) = \dim P$.*

COROLLARY 4. *$a \in F(P)$ is a stable face if and only if there exists a face b such that $a \perp b$ and $\dim(a \cup b) = \dim P$.*

3. Strong polytopes.

DEFINITION 4. A polytope P is said to be *strong* provided (i) if $a \in S(P)$ then $\{a\}^\perp$ is a principal order ideal in $F(P)$, (ii) if $a \perp b, a \not\perp\!\!\!\perp b$ where $a, b \in S(P)$ then $\{a, b\}^{\perp\!\!\!\perp}$ is a principal order ideal in $F(P) - \{1\}$.

If P is a strong polytope then $a \perp\!\!\!\perp b \iff a^\perp = b$ ($a \in S(P), b \in F(P)$), and therefore $S(P)$ is closed under the mapping $a \rightarrow a^\perp$.

THEOREM 5. *If P is a strong polytope then $\{S(P), \leq, \perp\}$ is an orthomodular poset. Its center coincides with the set of split faces. $\{S(P), \leq, \perp\}$ is an orthomodular lattice if and only if $\{a \vee b\}^\perp$ is a principle order ideal in $F(P)$ for all $a, b \in S(P)$.*

(For definitions and basic properties of orthomodular posets see [3].)

A simplex is a strong polytope and $S(P) = F(P) = \{\text{split faces}\}$. The orthomodular poset $\{S(P), \leq, \perp\}$ is Boolean (i.e.: a Boolean lattice).

THEOREM 6. *Let P be a strong polytope. If the orthomodular poset $\{S(P), \leq, \perp\}$ is Boolean then P is a simplex.*

4. States. The issue of this section is to show that with each point of a strong polytope P we can associate in a unique manner a state (probability measure) [2] on the orthomodular poset of stable faces. The treatment is purely geometrical.

Let P be a polytope with $\dim P \geq 1, S_1, S_2$ supports such that $S_1 \cap S_2 = \emptyset$ and $v \in \text{aff } P$. Then there exist elements $v_1 \in S_1, v_2 \in S_2$ and a unique real number $\mu(v, S_1, S_2)$ such that

$$v = \mu(v, S_1, S_2)v_1 + (1 - \mu(v, S_1, S_2))v_2.$$

Now let P be a strong polytope ($\dim P \geq 0$). For all $\omega \in P$ and $a \in S(P)$ we define

$$\mu_\omega(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } a = 1, \\ \mu(\omega, S_1, S_2) & \text{if } a \neq 0, 1, \end{cases}$$

where S_1, S_2 is the unique pair of supports such that $a \subseteq S_1$, $a^\perp \subseteq S_2$ and $S_1 \cap S_2 = \emptyset$ (see Theorem 2). Note that $0 \leq \mu_\omega(a) \leq 1$ for all $\omega \in P$ and $a \in S(P)$. Denote $\Delta = \{\mu_\omega \mid \omega \in P\}$.

THEOREM 7. *For every $\omega \in P$ the mapping $a \in S(P) \rightarrow \mu_\omega(a) \in [0, 1]$ is a state on the orthomodular poset $\{S(P), \leq, \perp\}$. The mapping $\omega \in P \rightarrow \mu_\omega \in \Delta$ is one-to-one.*

THEOREM 8. *Let P be a strong polytope. Then*

(i) Δ is a strongly order determining set of states for the orthomodular poset of stable faces;

(ii) μ_ω is a pure state (with respect to Δ) if and only if $\omega \in \text{ext } P$;

(iii) μ_ω is a dispersion-free state if and only if, for all $a \in S(P)$, $\omega \notin a$ implies $\omega \in a^\perp$;

(iv) μ_ω is a superposition [5] of a family of states $\{\mu_{\omega_i} \mid i \in I\}$ if and only if ω belongs to the face generated by $\{\omega_i \mid i \in I\}$.

5. Remark. In a subsequent paper we will give a characterization of those orthomodular posets that are ortho-order isomorphic to the orthomodular poset of stable faces of some strong polytope. The key notion in that investigation is a generalized version of the Jordan-Hahn decomposition of signed measures.

REFERENCES

1. E. M. Alfsen, *Compact convex sets and boundary integrals*, Springer-Verlag, Berlin and New York, 1971.
2. M. K. Bennett, *States on orthomodular lattices*, *Natur. Sci. and Math.* **8** (1968), 47–51. MR **38** #88.
3. D. J. Foulis, *A note on orthomodular lattices*, *Portugal. Math.* **21** (1962), 65–72. MR **26** #6088.
4. B. Grünbaum, *Convex polytopes*, *Pure and Appl. Math.*, vol. 16, Interscience, New York, 1967. MR **37** #2085.
5. V. S. Varadarajan, *Geometry of quantum theory*, Van Nostrand, Princeton, N. J., 1968.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MASSACHUSETTS,
AMHERST, MASSACHUSETTS 01002