BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 2, March 1976

ON THE PROBLEM OF EXIT[1]

BY B. J. MATKOWSKY AND Z. SCHUSS

Communicated by Hans Weinberger, December 10, 1975

We consider the effect of small random perturbations on a (deterministic) dynamical system $\dot{x} = b(x)$. The vector $x(t)$ then becomes a stochastic process $x_\epsilon(t)$. The perturbations are taken to be Gaussian white noise, i.e., $x_\epsilon(t)$ satisfies the stochastic differential equation

$$dx = b(x)\,dt + \epsilon\sigma(x)\,dw. \tag{1}$$

Here $w(t)$ is the n-dimensional Wiener process (Brownian motion), $b(x)$ is a vector field, $\sigma(x)$ is the diffusion matrix and $\epsilon \neq 0$ is a small real parameter.

The cumulative effect of even very small random perturbations may be considerable after sufficiently long times, so that even if the deterministic dynamical system has an asymptotically stable equilibrium point, the trajectories of the system will leave any compact domain with probability one. The following problem was posed by Kolmogorov: determine the probability distribution of the points on the boundary where trajectories exit, at the first time of their exit from a compact domain, as well as the expected exit times. The random effect may be thought of as a slow diffusion of particles in the deterministic flow field given by $b(x)$, and the results may differ according as particles are diffusing (a) with a flow, (b) across a flow, or (c) against a flow. Results on (a) were first obtained by Levinson [4], and on (b) by Khasminskiĭ [3], both of whom used analytical techniques. Problem (c) seems to be the most difficult, and to date only partial results are available (cf. Ventsel and Freidlin [5] and Friedman [1] who used probabalistic methods). Using analytical techniques, we present a full solution of this problem for flows which are essentially gradients of a potential (as well as certain more general flows).

Let Ω be a compact domain in R^n with a smooth boundary $\partial\Omega$. Let $a_{ij}(x) = \frac{1}{2}(\sigma(x)\sigma^*(x))_{ij}$, be strictly positive definite in $\overline{\Omega}$, $b(x) = (b_1, b_2, \ldots, b_n)$, and let $u_\epsilon(x)$ be the solution of the Dirichlet problem

$$L_\epsilon u \equiv \epsilon^2 \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} = g(x) \qquad (x \in \Omega), \tag{2}$$

$$u|_{\partial\Omega} = f(x).$$

AMS (MOS) subject classifications (1970). Primary 35B25, 60H10, 34F05.

[1] This research was supported in part by the National Science Foundation Grant MPS75-08328.

Copyright © 1976, American Mathematical Society

It is well known [1] that if $g \equiv 0$, then

$$u_\epsilon(x) = E_x f(x(\tau)),$$

and that if $g \equiv -1$ with $f \equiv 0$, then

$$u_\epsilon(x) = E_x \tau \sim \lambda_1^{-1}.$$

Here τ is the first exit time of a trajectory $x_\epsilon(t)$ of (1) from the domain Ω, E_x denotes the conditional expectation, given that $x_\epsilon(0) = x$ and λ_1 is the principal eigenvalue of L_ϵ. Let $0 \in \Omega$ denote the origin of the coordinate system and $\nu(x) = (\nu_1, \ldots, \nu_n)$ the outer normal to $\partial\Omega$. For any function $\psi(x)$ in $\overline{\Omega}$,

$$\frac{\partial \psi}{\partial n} \equiv \sum_{i,j=1}^{n} a_{ij}(x)\nu_i(x) \frac{\partial \psi}{\partial x_j} \qquad (x \in \partial\Omega),$$

represents its conormal derivative.

THEOREM 1. *Let $b(x)$ be a smooth vector field in $\overline{\Omega} - 0$, and assume*

$$b \cdot \nu = \sum_{i=1}^{n} b_i(x)\nu_i(x) < 0 \qquad (x \in \partial\Omega),$$

and

$$|b|^2 = \sum_{i=1}^{n} b_i^2(x) > 0 \qquad (x \in \overline{\Omega} - 0).$$

If there exists a function $\psi(x)$ in $\overline{\Omega}$ such that

$$b_j(x) = \sum_{i=1}^{n} a_{ij}(x) \frac{\partial \psi}{\partial x_i} \qquad (j = 1, 2, \ldots, n)$$

and if $g \equiv 0$, then $u_\epsilon(x) \longrightarrow c$, uniformly on any compact subset of Ω as $\epsilon \longrightarrow 0$. The constant c is given by

$$c = \lim_{\epsilon \to 0} \frac{\int_{\partial\Omega} \exp(\psi \epsilon^{-2})\rho(x) f(x)\, ds}{\int_{\partial\Omega} \exp(\psi\epsilon^{-2})\rho(x)\, ds}$$

with $\rho(x) = \partial\psi/\partial n = b \cdot \nu$.

THEOREM 2. *Let $p_\epsilon(y) = Pr\,(x_\epsilon(t) = y | x_\epsilon(0) = x)$ $(y \in \partial\Omega,\ x \in \Omega)$, be the probability density of the exit points of $x_\epsilon(t)$ from Ω. Then,* (i) *if the maximum of ψ on the boundary, is achieved on a set $U \subset \partial\Omega$, with nonempty interior U°, and that the measure of $U - U^\circ$ in $\partial\Omega$ is zero, then*

$$\lim_{\epsilon \to 0} p_\epsilon(y) = \begin{cases} \rho(y) \Big/ \int_{U^\circ} \rho(x)\, ds & \text{if } y \in U^\circ, \\ 0 & \text{if } y \in \partial\Omega - U^\circ; \end{cases}$$

(ii) *assume for simplicity that* $n = 2$ *and let* $(x, y) = (x(s), y(s))$ *be the parametric representation of* $\partial\Omega$, *where* s *denotes arc lenth on* $\partial\Omega$. *If the maximum of* ψ *on* $\partial\Omega$, *is achieved at the points* $s_1, s_2, \ldots, s_m$, *and*

$$\psi(x(s_i), y(s_i)) - \psi(x(s), y(s)) = d_i^{-2k_i}(s - s_i)^{2k_i}(1 + o(1))$$

as $s - s_i \longrightarrow 0$, *with* $d_i > 0$ $(i = 1, 2, \ldots, m)$, *and* $k \equiv k_1 = k_2 = \cdots = k_p = \max_i k_i$ $(p \leqslant m)$, *then*

$$\lim_{\epsilon \to 0} p_\epsilon(x(s), y(s)) = \sum_{j=1}^{p} \delta(s - s_j) d_j \rho(s_j) \Big/ \sum_{j=1}^{p} d_j \rho(s_j)$$

where $\delta(s)$ *is the Dirac measure on* $\partial\Omega$, *and* $\rho(s) = \rho(x(s), y(s))$.

THEOREM 3. *Let* $b(x)$ *and* $\psi(x)$ *be as in Theorem* 1, *with* ψ *normalized so that* $\psi(0) = 0$, *and assume that* $H(0) = \det \partial^2\psi/\partial x^2|_{x=0} \neq 0$. *If* $g(0) \neq 0$ (*e.g.*, $g \equiv -1$), *then*

$$u_\epsilon(x) \sim c(\epsilon)[1 - \exp(\zeta(x)/\epsilon^2)],$$

where

$$\zeta(x) = \rho(x)\mathrm{dist}(x, \partial\Omega) \Big/ \sum_{i,j=1}^{n} a_{ij}\nu_i(x)\nu_j(x).$$

In case (i) of Theorem 2,

$$c(\epsilon) = \frac{(2\pi\epsilon^2)^{n/2} g(0) \exp(-\hat{\psi}/\epsilon^2)}{H^{1/2}(0) \int_{U^\circ} \rho(x)\, ds}$$

with $\hat{\psi} = \max_{\partial\Omega} \psi < 0$.

In case (ii) of Theorem 2

$$c(\epsilon) = \frac{2\pi k \epsilon^{(2k-1)/k} g(0) \exp(-\hat{\psi}/\epsilon^2)}{\Gamma(1/2k) H^{1/2}(0) \Sigma_{j=1}^{p} d_j \rho(s_j)}$$

Note that the expected exit time increases exponentially and the principal eigenvalue decreases exponentially, as $\epsilon \longrightarrow 0$.

REMARKS. (I) Similar results hold for higher dimensions in (ii) as well as for more general equations in all Theorems, and also if $g(0) = 0$ and/or $H(0) = 0$.

(II) The results of Ventsel and Freidlin correspond to $m = 1$ in (ii), for somewhat more general flows, while the results of Friedman for Theorem 1, correspond to $f =$ constant on the set of maxima.

(III) The methods we employ are generalizations of the method developed by Grasman and Matkowsky [2].

REFERENCES

1. A. Friedman, *Stochastic differential equations*, Academic Press, New York, 1975.

2. J. Grasman and B. J. Matkowsky, *A variational approach to boundary layer problems and resonance for ordinary and partial differential equations*, SIAM J. Appl. Math. (submitted).

3. R. Z. Has′minskiĭ, *Diffusion processes with a small parameter,* Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), 1281–1300. (Russian) MR **29** #6530.

4. N. Levinson, *The first boundary value problem for* $\epsilon\Delta u + A(x, y)u_x + B(x, y)U_y + C(x, y)u = D(x, y)$ *for small* ϵ, Ann. of Math. (2) **51** (1950), 428–445. MR **11**, 439.

5. A. D. Ventcel′ and M. I. Freĭdlin, *On small perturbations of dynamical systems,* Uspehi Mat. Nauk **25** (1970), no. 1 (151), 3–55 = Russian Math. Surveys **25** (1970), no. 1, 1–56. MR **42** #2123.

DEPARTMENT OF MATHEMATICS, RENSSELAER POLYTECHNIC INSTITUTE, TROY, NEW YORK 12181